

On the Convergence of Fourier Series and Transforms.

Reference: E. C. Titchmarsh, *The Theory of Functions*, 2nd. ed., Chapter 13. (Much of this handout is stolen verbatim from this reference.)

In some ways I feel the discussions in Sections 3.4 and 4.1.2 of OW2 are unsatisfactory, and the object of these notes is to give a brief glimpse into the fascinating mathematics of the convergence of the Fourier series and transforms.

1. Fourier Series for Periodic Functions.

In what follows, for simplicity, I shall discuss a function which is periodic of period $T = 2\pi$, but the results stated do not depend in any essential way on the exact value of T .

Suppose then that $f(t)$ is a function defined on the interval $0 \leq t < 2\pi$, and outside this interval we define it by periodicity, i.e.,

$$f(t + 2\pi) = f(t).$$

The Fourier coefficients of $f(t)$ are defined by

$$(1) \quad a_k \triangleq \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-jkt} dt, \quad \text{for } k = 0, \pm 1, \pm 2, \dots$$

We cannot even begin our studies unless the integrals in (1) exist. Luckily, it can be shown that if the “DC” coefficient a_0 exists, then all the rest do, too, so our first assumption about $f(t)$ is that

$$(2) \quad f(t) \text{ is integrable on } (0, 2\pi).$$

(Technically, we mean *Lebesgue* integrable.) Note, for example, that the function shown in Figure 3.8 (a) in OW2 is not integrable, so its Fourier series does not even exist, let alone converge.

But if (2) is satisfied, the coefficients a_k in (1) all exist, so we can form the series

$$(3) \quad \sum_{k=-\infty}^{\infty} a_k e^{jkt},$$

and study its convergence properties. It turns out that the series does substantially what is required of it. We should not, however, expect too much.

It is clearly impossible that, whatever $f(t)$ is, the series should converge to $f(t)$ for every value of t . Consider, for example, two functions $f(t)$ and $g(t)$ which differ at one point only. Then $f(t)$ and $g(t)$ have the same Fourier series, which cannot therefore represent both functions at every point. More generally, two functions which are equal “almost everywhere” have the same Fourier series, which therefore cannot represent them both if they differ anywhere.

What is true, however, is that the series does represent the function, provided that the function is not too complicated; and even in the most complicated cases, the series still represents in some sense the main features of the function. In the remainder of this handout we shall summarize some of what is known.

Let $0 \leq t < 2\pi$ and let $s_n(t)$ denote the n th partial sum of (3), i.e.,

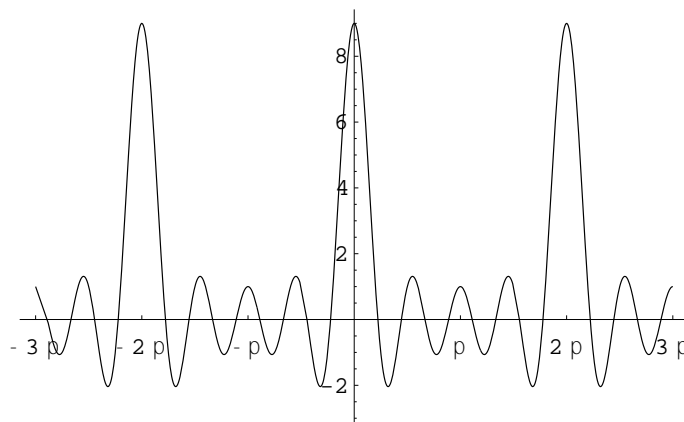
$$(4) \quad s_n(t) \triangleq \sum_{k=-n}^n a_k e^{jkt}.$$

A reasonably simple calculation, which we omit, shows that in fact

$$(5) \quad s_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-u) D_n(u) du,$$

where $D_n(u)$ is the *first Dirichlet kernel*, defined by

$$(6) \quad D_n(u) \triangleq \frac{\sin(n + \frac{1}{2})u}{\sin \frac{1}{2}u}.$$



The Dirichlet kernel $D_4(u)$.

For example, if $f(t) = 1$ for all t , then $a_0 = 1$ and all the rest of the Fourier coefficients are zero, so that $s_n(t) = 1$ for all $n > 0$. In this case (5) becomes

$$1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(u) du.$$

Multiplying this by s and subtracting from (5), we have

$$s_n(t) - s = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(t-u) - s) D_n(u) du.$$

We have therefore proved a basic tool of the subject:

Theorem 1. *A necessary and sufficient condition that the series in (3) should converge to the sum s is that*

$$(7) \quad \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(t-u) - s) D_n(u) du = 0.$$

Theorem 1, in the hands of competent mathematicians, has led to many sufficient conditions for the series (3) to converge to $f(t)$. As a start towards these results, we observe that for any $\delta > 0$,

$$\lim_{n \rightarrow \infty} D_n(u) = 0 \quad \text{for } |u| > \delta,$$

which implies¹ that the condition (7) can be replaced by

$$(8) \quad \lim_{n \rightarrow \infty} \int_{-\delta}^{\delta} (f(t-u) - s) D_n(u) du = 0,$$

for any $\delta > 0$. The alternative formulation (8) is very enlightening, since among other things it tells us that *the convergence or nonconvergence of the Fourier series to $f(t_0)$ depends only on the behavior of $f(t)$ in the local vicinity of $t = t_0$.*

For example, for a fixed value of t , let

$$\phi(u) = f(t-u) - s.$$

Then it is possible to prove “Dini’s test”:

¹ Actually, this step, which seems straightforward enough, requires a short auxiliary result which has come to be called the *Riemann-Lesbeque lemma*.

Dini's Test. If $\phi(u)/u$ is integrable over the interval $(-\delta, \delta)$, then the series in (3) converges to s .

If, for example, $f(t)$ is differentiable at t , then with $s = f(t)$, we have

$$\frac{\phi(u)}{u} = \frac{f(t-u) - f(t)}{u},$$

which approaches $f'(t)$ as $u \rightarrow 0$, so that in a small interval $(-\delta, \delta)$ with small positive δ , $\phi(u)/u$ is bounded and so integrable. Thus we have proved

Corollary to Dini's Test. If $f(t)$ is differentiable at t , then the series (3) converges to $f(t)$.

Note that in particular, the function depicted in Figure 3.8 (b), p. 199 OW2, is everywhere differentiable, and so its Fourier series converges to $f(t)$ everywhere, even though the function violates the second Dirichlet condition!

Another important test is "Jordan's test:"

Jordan's Test. If $f(t)$ is of bounded variation in any neighborhood of t , then the series $s_n(t)$ converges to the sum $(f(t+0) + f(t-0))/2$.²

"Bounded variation" is a technical term, which we do not define precisely here. Roughly speaking, though, a function $y = f(t)$ defined on an interval (a, b) is said to be of bounded variation on (a, b) if the total change in y as t varies from a to b (with positive and negative changes both counted positively, so there is no cancellation of variation) is finite. For example, the function shown in Figure 3.8 (b) in OW2 is of unbounded variation, since it goes up and down with a fixed amplitude infinitely often, but it is of bounded variation in a neighborhood of any point but $t = 0$. Thus by Jordan's test, the Fourier series for the function shown in Figure 3.8(b) converges to $f(t)$ for every t except $t = 0$. The function shown in Figure 3.8 (c) has bounded variation (in fact, since it is monotonic, its total variation in the given interval is 1). Thus by Jordan's test, the Fourier series for the function shown in Figure 3.8 (c) converges to $f(t)$ for every value of t for which $f(t)$ is continuous, even though the function violates the third Dirichlet condition!

According to Jordan's test, if $f(t)$ is of bounded variation in a small interval containing t , then series converges at t . But its value represents not $f(t)$ itself, but rather a sort of limit of the average value of $f(t)$ over the interval $(t - \delta, t + \delta)$, which will be equal to

² The notation $f(t+0)$ is shorthand for the limit of $f(t+h)$ as h approaches zero through positive values, and $f(t-0)$ is the limit as h approaches zero through negative values.

$f(t)$ only if the behavior of the function is sufficiently simple. And as we have already remarked, the value of the function at t itself does not determine or affect in any way the sum of the series.

How then do the three Dirichlet Conditions cited in OW2, Section 3.4, fit in? It turns out that *a function with only a finite number of maxima and minima and a finite number of discontinuities in the interval $(0, 2\pi)$ must be of bounded variation in $(0, 2\pi)$* , and so by Jordan's test, $s_n(t)$ converges to $(f(t+0) + f(t-0))/2$ for all $t \in (0, 2\pi)$. In short, the first Dirichlet condition is just a restatement of the fundamental assumption (2), and the second two Dirichlet conditions are just a "brute-force" way of guaranteeing that $f(t)$ is of bounded variation in $(0, 2\pi)$. Thus:

The Dirichlet conditions are just a poor man's version of Jordan's test.

Finally we may ask, in view of the tests of Dini and Jordan, if there are any integrable functions at all for which the Fourier series in (3) behaves unexpectedly? The answer is yes, but it takes a twisted and brilliant mathematical mind to dream one up. In Chapter 13 of Titchmarsh, as cited above, there is constructed a continuous, integrable, function $f(t)$ on $(0, 2\pi)$, whose Fourier series diverges at $t = 0$. This famous example was devised by L. Fejér in 1910.

2. Fourier Transforms for Non-Periodic Functions.

In this section, we suppose that $f(t)$ is function defined for all real values of t , and that

$$(9) \quad f(t) \text{ is integrable on } (-\infty, +\infty).$$

Then we are justified in defining the *Fourier transform* of $f(t)$ as

$$(10) \quad F(\omega) \triangleq \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt.$$

The *inverse Fourier transform* of $F(\omega)$ is the function of t defined by

$$(11) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega.$$

The problem is whether or not the integral in (11) converges to $f(t)$. As with the Fourier series, we shall see that the inverse Fourier transform does substantially what we would expect, unless the function is incredibly badly behaved.

To study this convergence problem, we define, for any positive number L , the integral

$$(12) \quad I_L(t) \triangleq \frac{1}{2\pi} \int_{-L}^L F(\omega)e^{j\omega t} d\omega.$$

The goal, of course, is to study the behavior of $I_L(t)$ as $L \rightarrow \infty$. To do so, we replace the function $F(\omega)$ which appears in (12) with its definition (10) and interchange the order of integration:³

$$\begin{aligned}
 I_L(t) &= \frac{1}{2\pi} \int_{-L}^L \left(\int_{-\infty}^{\infty} f(v) e^{-j\omega v} dv \right) e^{j\omega t} d\omega \\
 (13) \qquad &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(v) \left(\int_{-L}^L e^{-j\omega(v-t)} d\omega \right) dv.
 \end{aligned}$$

Now substitute the variable $u = t - v$ in (13). The result is

$$(14) \qquad I_L(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t - u) \left(\int_{-L}^L e^{j\omega u} d\omega \right) du.$$

It is an easy exercise to show that

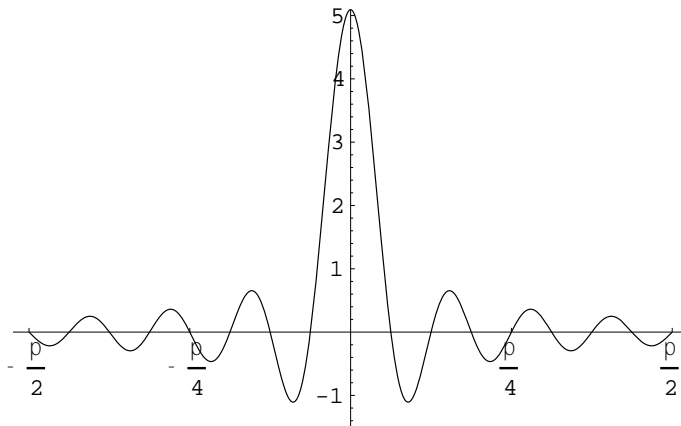
$$\int_{-L}^L e^{j\omega u} d\omega = 2 \frac{\sin Lu}{u},$$

so that (14) becomes

$$(15) \qquad I_L(t) = \int_{-\infty}^{\infty} f(t - u) K_L(u) du,$$

where $K_L(u)$ is the *second Dirichlet kernel*, defined by

$$(16) \qquad K_L(u) \triangleq \frac{1}{\pi} \frac{\sin Lu}{u} = \frac{L}{\pi} \operatorname{sinc} \left(\frac{Lu}{\pi} \right).$$



The Dirichlet kernel $K_{16}(u)$.

³ This procedure is justified because the integral for $F(\omega)$ converges uniformly with respect to ω .

It is easy to see that

$$(17) \quad \int_{-\infty}^{\infty} K_L(u)du = 1,$$

so that if we multiply (17) by s and subtract it from (15), we arrive at another basic tool.

Theorem 2. *A necessary and sufficient condition that the inverse Fourier transform (11) converges to the value s is that*

$$(18) \quad \lim_{L \rightarrow \infty} \int_{-L}^L (f(t-u) - s)K_L(u)du = 0.$$

By reasoning similar to that which led from (7) to (8), we find that a condition equivalent to (7) is

$$(19) \quad \lim_{L \rightarrow \infty} \int_{-\delta}^{\delta} (f(t-u) - s)K_L(u)du = 0,$$

for some positive value of δ .

It follows then by reasoning similar to that which led to Dini's test and Jordan's test, that we have the following two results.

Corollary 1. *If $f(t)$ is differentiable at $t = t_0$, then the inverse Fourier transform (11) converges to $f(t)$.*

Corollary 2. *If $f(t)$ is of bounded variation in an interval containing $t = t_0$, then the inverse Fourier transform (11) converges to $(f(t+0) + f(t-0))/2$.*