EE32a Signals, Systems, and Transforms — R. J. McEliece
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Some ROC Theory for Laplace Transforms

In this handout we will present (with modest justification) formulas which describe the regions of convergence for Laplace transforms. This material closely parallels, but is not identical to, the material in OW2, Section 9.2.

1. The Laplace Transform.

If $x(t)$ is a continuous-time signal, its *Laplace transform* is the function $X(s)$ of the complex variable s defined as follows:

$$X(s) = \int_{-\infty}^{\infty} e^{-st} x(t)dt. \tag{1}$$

We wish to know when the integral in (1) converges. Recall from calculus that by definition the integral in (1) converges if and only if each of the following two limits exists:

$$X_R(s) = \lim_{T_1 \to \infty} \int_0^{T_1} e^{-st} x(t)dt. \tag{2}$$

$$X_L(s) = \lim_{T_2 \to \infty} \int_{-T_2}^{0} e^{-st} x(t)dt = \lim_{T_2 \to \infty} \int_0^{T_2} e^{st} x(-t)dt, \tag{3}$$

in which case, of course,

$$X(s) = X_R(s) + X_L(s).$$

We call $X_R(s)$ the *right part*, and $X_L(s)$, the *left part*, of $X(s)$.

The main result about the convergence of $X(s)$ is the following.

Theorem 1. *There exist two real numbers σ_R and σ_L (possibly $\pm\infty$), such that: If $\sigma_R < \mathrm{Re}(s) < \sigma_L$, then $X(s)$ converges absolutely,*[1] *and in fact $X(s)$ is an analytic function in this region. On the other hand, if $\mathrm{Re}(s) < \sigma_R$ or if $\mathrm{Re}(s) > \sigma_L$, then $X(s)$ diverges. This result is summarized by saying that $\{s : \sigma_R < \mathrm{Re}(s) < \sigma_L\}$ is the region of convergence for the Laplace transform $X(s)$.*

The numbers σ_L and σ_R are called the (left and right) *abscissas of convergence* for $X(s)$. In the most general case, they are somewhat difficult to compute, but the following result suffices for most cases.

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[1] By this we mean that $\int |e^{-st}x(t)|dt = \int e^{-\sigma t}|x(t)|dt$, where $\sigma - \mathrm{Re}(s)$, converges. This condition is stronger than ordinary convergence.

Theorem 2. *With σ_L and σ_R as in Theorem 1, we have*

$$\sigma_R = \lim_{t\to\infty} \frac{1}{t} \log |x(t)| \tag{4}$$

$$\sigma_L = \lim_{t\to-\infty} \frac{1}{t} \log |x(t)|, \tag{5}$$

provided these limits exist.

Proof: We will cheat by invoking a theorem from complex variable theory that asserts that a “one-sided” Laplace transform of the form

$$F(s) = \int_0^\infty e^{-st} f(t) dt \tag{6}$$

converges absolutely for $\text{Re}(s) > \sigma$ and diverges for $\text{Re}(s) < \sigma$, where the number σ (the abscissa of convergence) is given by the formula

$$\sigma = \lim_{t\to\infty} \frac{1}{t} \log |f(t)|, \tag{7}$$

provided this limit exists. (If the limit in (7) does not exist, this theorem does not apply.) (A brief sketch of a proof of this result appears in Appendix A of this handout.)

If we take this result for granted, Theorem 2 follows quite easily. The right part $X_R(s)$ of $X(s)$ defined in (2) is a one-sided Laplace transform, and so by (7), it converges absolutely for $\text{Re}(s) > \sigma_R$ and diverges for $\text{Re}(s) < \sigma_R$, where σ_R is defined in (4). The left part $X_L(s)$ defined in (3) is also a one-sided Laplace transform(in the complex variable $-s$), and so by (7), its region of convergence is $\text{Re}(-s) > \sigma_0$, where σ_0 is defined by the formula (cf. (7))

$$\sigma_0 = \lim_{t\to\infty} \frac{1}{t} \log |x(-t)| \tag{8}$$

$$= \lim_{t\to-\infty} -\frac{1}{t} \log |x(t)| \tag{9}$$

But the condition $\text{Re}(-s) > \sigma_0$, with σ_0 as defined in (8) is identical to the condition $\text{Re}(s) < \sigma_L$, where σ_L is as defined in (5). Thus the region of convergence for $X_l(s)$ is $\text{Re}(s) < \sigma_L$. Since we have already seen that the region of convergence for $X_R(s)$ is $\text{Re}(s) > \sigma_R$, this completes the proof of Theorem 3. ▪

Example 1. Suppose that $x(t)$ is given by

$$x(t) = \begin{cases} e^{-2t+\sqrt{t}} & \text{if } t \geq 0 \\ te^{3t} & \text{if } t < 0. \end{cases}$$

Then by (4) and (5) we have $\sigma_R = -2$ and $\sigma_L = 3$, so that the ROC for this signal is $-2 < |z| < 3$. An interesting thing about this example is that we are able to calculate the ROC even though we have no closed-form expression for $X(s)$ itself. ▪

Example 2. Suppose $x(t) = e^{-at}u(t)$, where a ia a real number. Then by (4) and (5), we have

$$\sigma_R = \lim_{t\to\infty} \frac{1}{t}\log 1 = -a$$
$$\sigma_L = \lim_{t\to-\infty} \frac{1}{t}\log 0 = \infty.$$

so that the ROC for $u(t)$ is $\{s : \mathrm{Re}(s) > -a\}$.[2] In this case a short calculation shows that in fact, if $s \in$ ROC,

$$X(s) = \frac{1}{s+a}. \qquad ▪$$

Example 3. Now suppose $x(t) = -e^{-at}u(-t)$. In this case we have

$$\sigma_R = \lim_{t\to\infty} \frac{1}{t}\log 0 = -\infty$$
$$\sigma_L = \lim_{t\to-\infty} \frac{1}{t}\log 1 = 0.$$

so that the ROC for $-u(-t)$ is $\{s : \mathrm{Re}(s) < 0\}$. In this case a short calculation shows that in fact, if $s \in$ ROC,

$$X(s) = \frac{1}{s+a},$$

the same as in Example 2. ▪

Example 4. Let $x(t)$ be any signal of *finite duration*, i.e., $x(t) = 0$ unless $t_0 < t < t_1$. Then it is easy to see that $\sigma_R = -\infty$ and $\sigma_L = \infty$, i.e., $X(s)$ converges for all s. ▪

Example 5. Let $x(t) = e^{t^2}u(t)$. Then by (4) and (5), we have

$$\sigma_R = \lim_{t\to\infty} \frac{1}{t}\log e^{t^2} = \infty$$
$$\sigma_L = \lim_{t\to-\infty} \frac{1}{t}\log 0 = \infty.$$

so that the ROC for $u(t)$ is the empty set! ▪

Example 6. This example will begin to hint at the complications the Laplace transform can enjoy. Let $x(t)$ be defined by

$$x(t) = e^{t\sin t}u(t).$$

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[2] I admit that the calculation of σ_L is this cae is somewhat bogus. If you don't like the derivation, then just take it as given that $\sigma_L = \infty$ for any *right-sided* signal, i.e., one for which $x(t) = 0$ for all sufficiently small values of t. Similarly, for a left-sided signal, i.e., one for which $x(t) = 0$ for all sufficiently large t, we have $\sigma_R = -\infty$.

Since $x(t)$ is a right-sided, we have $\sigma_L = \infty$. Attempting to compute σ_R using (4), we find

$$\begin{aligned}\sigma_R &= \lim_{t\to\infty} \frac{1}{t} \log e^{t \sin t} \\ &= \lim_{t\to\infty} \sin t \\ &= ???\end{aligned}$$

We will return to Example 6 in Appendix B. ▪

2. Stability of Continuous-Time LTI systems.

It is normally possible to determine whether or not a continuous-time LTI system is stable by knowing the ROC. For according to OW2, Section 2.3.7, a system with impulse response $h(t)$ is stable if and only if $h(t)$ is absolutely integrable, i.e., if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty. \tag{10}$$

The condition (10) is equivalent to saying that the Laplace transform $H(s)$ of $h(t)$ is *absolutely convergent* on the imaginary axis, i.e., the line $\{s : \mathrm{Re}(s) = 0\}$. We know from Theorem 3 that $H(s)$ is absolutely convergent in the open ROC and diverges outside the closed ROC, where

$$\begin{aligned}\text{open ROC} &= \{s : \sigma_R < \mathrm{Re}(s) < \sigma_L\} \\ \text{closed ROC} &= \{s : \sigma_R \le \mathrm{Re}(s) \le \sigma_L\}.\end{aligned} \tag{11}$$

Thus we have the following.

Theorem 3. *If the open ROC for $H(s)$ contains the point $s = 0$, then the system is stable. If the point $s = 0$ lies outside the closed ROC, then the system is unstable. On the other hand, if $s = 0$ lies on one of the critical lines no general conclusion is possible.* ▪

Theorem 4. *Suppose $H(s)$ is rational. Then the corresponding system is stable if and only if the line $\{s : \mathrm{Re}(s) = 0\}$ is contained in the open ROC.*▪

Theorem 5. *Suppose $H(s)$ is rational and causal. Then the corresponding system is stable if and only if $H(s)$ has no poles in the region $\{s : \mathrm{Re}(s) \ge 0\}$, which is sometimes called "the right half plane."* ▪

Appendix A. Sketch of Proof of Theorem 4.

If the limit in (6) exists, then for large values of t, $|f(t)|$ is given asymptotically by the formula

$$|f(t)| \sim e^{\sigma t + o(t)}, \tag{12}$$

where "$o(t)$" (pronounced "little oh of t"), is a function that grows more slowly than any constant times t. Thus if $\mathrm{Re}(s) = x$, we have

$$\int_0^\infty |e^{-st} f(t)| dt = \int_0^\infty e^{-xt} |f(t)| \tag{13}$$

$$= \int_0^\infty e^{-t(x-\sigma)+o(t)} dt \tag{14}$$

which plainly converges if $x > \sigma$ and diverges if $x < \sigma$. ▪

Appendix B. A Better Description of the ROC for Laplace transforms.

Better, but more complicated. It deals with a one sided Laplace transform of the form (6). Reference: Widder, *The Laplace Transform*, Section II.2.

Theorem W. *Assume that*

$$\int_0^R |f(t)| dt$$

exists for each $R > 0$. Then the abscissa of convergence is given by the formula

$$\sigma = \begin{cases} \limsup_{t\to\infty} \frac{1}{t} \log \left| \int_0^t f(\tau) d\tau \right| & \text{if } \int_0^\infty f(\tau) d\tau \text{ diverges} \\ \limsup_{t\to\infty} \frac{1}{t} \log \left| \int_t^\infty f(\tau) d\tau \right| & \text{if } \int_0^\infty f(\tau) d\tau \text{ converges.} \end{cases} \tag{15}$$

Example 6. (continued) We return to Example 6, in the light of Theorem W. Clearly in this case $\int_0^\infty f(\tau) d\tau$ diverges, so that by (15), the abscissca of convergence is

$$\sigma = \limsup_{t\to\infty} \left(\frac{1}{t} \log \int_0^t e^{\tau \sin \tau} d\tau \right)$$
$$= ??$$