Structures for Discrete-Time LTI Systems.

1. The Examples.

In these notes, we will sketch three ways to represent a structure for a causal discrete-time system which can be described by a finite difference equation:

- A block diagram.
- A signal flow graph.
- An adjacency matrix.

We begin with four examples (the block diagram and flow graphs are given in the figures, and the adjacency matrices are given in the text).

Example 1. A causal LTI system, represented by a block diagram and a signal flow graph, with system function $H(z) = p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}$. (In the signal flow graph, the input node is 1 and the output node is 0.)
For the signal flow graph in Example 1, the adjacency matrix is

\[
A_1 = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
1 & p_0 & D & & \\
2 & p_1 & D & & \\
3 & p_2 & D & & \\
4 & p_3 & & & \\
\end{pmatrix}
\]

(Here and hereafter, missing entries in \( A \) are assumed to be zero.)

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**Example 2.** A causal LTI system, represented by a block diagram and a signal flow graph, with

\[
H(z) = \frac{1}{1 + q_1 z^{-1} + q_2 z^{-2} + q_3 z^{-3}}.
\]

(In the signal flow graph, the input node is 1 and the output node is also 1.)

For the signal flow graph in Example 2, the adjacency matrix is

\[
A_2 = \begin{pmatrix}
1 & 2 & 3 & 4 \\
2 & -q_1 & D & & \\
3 & -q_2 & D & & \\
4 & -q_3 & & & \\
\end{pmatrix}
\]
Example 3. A causal LTI system, represented by a block diagram and a signal flow graph, with

$$H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}}{1 + q_1 z^{-1} + q_2 z^{-2} + q_3 z^{-3}}.$$  

(In the signal flow graph, the input node is 1 and the output node is 0.) This structure is sometimes called the *controller canonical form.*
For the signal flow graph in Example 3, the adjacency matrix is

\[
A_3 = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
1 & p_0 & D & & \\
2 & p_1 & -q_1 & D & \\
3 & p_2 & -q_2 & D & \\
4 & p_3 & -q_3 & & \\
\end{pmatrix}
\]

Example 4. A causal system, represented by a block diagram and a signal flow graph, with

\[
H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}}{1 + q_1 z^{-1} + q_2 z^{-2} + q_3 z^{-3}}.
\]

(In the signal flow graph, the input node is 0 and the output node is 1.) This structure is sometimes called the observer canonical form. It is the transpose of the controller canonical form shown in Example 3.
For the signal flow graph in Example 4, the adjacency matrix is

\[
A_4 = A_3^T = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & p_0 & p_1 & p_2 & p_3 \\
1 & -q_1 & -q_2 & -q_3 & D \\
2 & D & D \\
3 & D \\
4 & D
\end{pmatrix}
\]

2. The Transfer Function Theorem.

Given an LTI system as represented by a flowgraph, denote by \(H_{i,j}(z)\) the system function if vertex \(i\) is selected as the input and vertex \(j\) is selected as the output.

The main theorem, whose proof was given in class, is this.

**Theorem.** We have

\[
H_{i,j}(z) = \left[ (I_N - A)^{-1} \right]_{i,j} = \frac{(-1)^{i+j} \det(I_N - A; j, i)}{\det(I_N - A)}.
\]

(The notation \((M : i, j)\) signifies the matrix \(M\) with row \(i\) and column \(j\) deleted.)

For example, consider Example 3. Here the problem is to find the \((1,0)\)th entry of the inverse of the matrix

\[
I_5 - A_3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-p_0 & 1 & -D & 0 & 0 \\
-p_1 & q_1 & 1 & -D & 0 \\
-p_2 & q_2 & 0 & 1 & -D \\
-p_3 & q_3 & 0 & 0 & 1
\end{pmatrix}.
\]
By the Cofactor Theorem, this entry is

\[ - \det \begin{pmatrix} -p_0 & D & 0 & 0 \\ -p_1 & 1 & -D & 0 \\ -p_2 & 0 & 1 & -D \\ -p_3 & 0 & 0 & 1 \end{pmatrix} \]

\[ \frac{\det \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ -p_0 & 1 & -D & 0 \\ -p_1 & q_1 & 1 & -D \\ -p_2 & q_2 & 0 & 1 & -D \\ -p_3 & q_3 & 0 & 0 & 1 \end{array} \right)} {\det \left( \begin{array}{ccc} 1 & -D & 0 \\ q_1 & 1 & -D \\ q_2 & 0 & 1 & -D \\ q_3 & 0 & 0 & 1 \end{array} \right)} \]

\[ \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}} {1 + q_1 z^{-1} + q_2 z^{-2} + q_3 z^{-3}} \]

Finally note that no further computation is required to verify the correctness of the alleged system function in Example 4, since \( A_4 \) is the transpose of \( A_3 \) and hence has the same determinant and cofactors.