Problem 1. (Graded by Muhammed) The first step is to decompose $X(s)$ into partial fractions:

$$\frac{s - 1}{(s + 2)(s + 3)(s^2 + s + 1)} = \frac{-1}{s + 2} + \frac{4/7}{s + 3} + \frac{(3s + 1)/7}{s^2 + s + 1} = X_1(s) + X_2(s) + X_3(s).$$

Note that the poles of $X(s)$ are at $s = -2, -3, \text{ and } -1/2 \pm j\sqrt{3}/2$.

The inverse Laplace transforms of $X_1(s)$, $X_2(s)$, and $X_3(s)$ are

$$x_1(t) = \begin{cases} x_1^R(t) = -e^{-2t}u(t) & \text{Re}(s) > -2 \\ x_1^L(t) = e^{-2t}u(-t) & \text{Re}(s) < -2 \end{cases}$$

$$x_2(t) = \begin{cases} x_2^R(t) = (4/7)e^{-3t}u(t) & \text{Re}(s) > -3 \\ x_2^L(t) = -(4/7)e^{-3t}u(-t) & \text{Re}(s) > -3 \end{cases}$$

$$x_3(t) = \begin{cases} x_3^R(t) = \frac{1}{7} \left( 3e^{-t/2} \cos(\sqrt{3}t/2) - \frac{1}{\sqrt{3}}e^{-t/2} \sin(\sqrt{3}t/2) \right) u(t) & \text{Re}(s) > -1/2 \\ x_3^L(t) = -\frac{1}{7} \left( 3e^{-t/2} \cos(\sqrt{3}t/2) - \frac{1}{\sqrt{3}}e^{-t/2} \sin(\sqrt{3}t/2) \right) u(-t) & \text{Re}(s) < -1/2 \end{cases}.$$

The four inverse Laplace transforms of the original $X(s)$ are then

- $\text{Re}(s) < -3 : x_1^L(t) + x_2^L(t) + x_3^L(t)$
- $-3 < \text{Re}(s) < -2 : x_1^L(t) + x_2^R(t) + x_3^L(t)$
- $-2 < \text{Re}(s) < -1/2 : x_1^R(t) + x_2^R(t) + x_3^L(t)$
- $\text{Re}(s) > -1/2 : x_1^R(t) + x_2^R(t) + x_3^R(t)$

$$x(t), \text{Re}(s) < -3.$$
$x(t), -3 < \text{Re}(s) < -2.$

$\text{Re}(s) < -1/2.$
\( x(t), \Re(s) > -1/2. \)
Problem 2. (Graded by all.)

(a) Use the fact that $\text{Re } x(t) = (x(t) + x^*(t))/2$, and entry 4.3.3 in Table 4.1, p. 328:

$$\text{Re } x(t) \xrightarrow{\mathcal{F}} \frac{X(j\Omega) + X^*(-j\Omega)}{2}.$$ 

(b) Using entries 4.3.3 and 4.3.5 in Table 4.1:

$$x^*(-t) \xrightarrow{\mathcal{F}} X^*(j\Omega).$$

(c) Using the given decomposition and entry 4.3.5 in Table 4.1:

$$\text{Ev } x(t) \xrightarrow{\mathcal{F}} \frac{X(j\Omega) + X(-j\Omega)}{2}.$$
Problem 3. (Graded by RJM) The key to this problem is to use $z$-transforms. Indeed, if $g[n] * g[n] = h[n]$, we have

$$G(z)^2 = H(z).$$

Thus if $g[n]$ is a convolution square root of $h[n]$, we have

$$G(z) = \pm \sqrt{H(z)}.$$

(a) Here $H(z) = 1 + 2z^{-1} + z^{-2} = (1 + z^{-1})^2$, so $G(z) = \pm(1 + z^{-1})^2$. Ignoring the possible minus sign, we have

$$g[n] = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 1 \\ 0 & \text{otherwise}. \end{cases}$$

(b) Here we have

$$H(z) = \sum_{n \geq 0} (n + 1)z^{-n} = \frac{1}{(1 - z^{-1})^2}.$$  

Thus $G(z) = 1/(1 - z^{-1})$, so that (again ignoring the possible minus sign)

$$h[n] = u[n].$$

(c) Here $H(z) = \sum_{n \geq 0} z^{-1} = 1/(1 - z^{-1})$, so that $G(z) = (1 - z^{-1})^{-1/2}$. But as we discussed in class on Nov. 9, we can apply the binomial theorem, even when the exponent is negative, so that

$$g(z) = (1 - z^{-1})^{-1/2} = \sum_{n \geq 0} (-1)^n \binom{-1/2}{n} z^{-n}.$$  

Thus

$$g[n] = (-1)^n \binom{-1/2}{n} u[n].$$

This expression can be simplified considerably:

$$(-1)^n \binom{-1/2}{n} = (-1)^n \cdot \frac{(-1/2)(-3/2) \cdots (-2n-1/2)}{1 \cdot 2 \cdots n} = \binom{1/2}{n} \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n}. $$

Thus $g[0] = 1$, and for $n \geq 1$,

$$g[n] = \prod_{k=1}^{n} \left( 1 - \frac{1}{2k} \right).$$

For example $g[1] = 1/2$, $g[2] = 3/8$, $g[3] = 5/16$, etc.

Notes: A few students used Mathematica to find the inverse $z$-transform of $G(z)$, and obtained the remarkable expression

$$g[n] = \frac{\Gamma(n + 1/2)}{\Gamma(n + 1)\sqrt{\pi}}.$$  

I reluctantly accepted this formula. Also, some students found a recursive method for computing the sequence of $g[n]$’s. I took off 2.5 points for this approach.
Problem 4. (Graded by Donald) This system in the serial iterconnection of two simpler systems, with systems functions (obtained by “inspection,” perhaps)

\[ H_1(D) = \frac{1 - \frac{1}{2}D}{1 - \frac{1}{3}D}, \]
\[ H_2(D) = \frac{\frac{1}{4} + D}{1 + \frac{1}{2}D}, \]

where I have substituted the delay operator \( D \) for the cumbersome \( z^{-1} \). Thus the overall system function is

\[ H(D) = H_1(D)H_2(D) = \frac{\frac{1}{4} + \frac{7}{8}D - \frac{1}{2}D^2}{1 + \frac{1}{6}D - \frac{1}{6}D^2}. \]

from this we can read off the difference equation. It is

(a) \[ y[n] + \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = \frac{1}{4}x[n] + \frac{7}{8}x[n-1] - \frac{1}{2}x[n-2]. \]

Of course the frequency response is just the systems function evaluated at \( z = e^{j\Omega} \):

(b) \[ H(e^{j\Omega}) = \frac{\frac{1}{4} + \frac{7}{8}e^{-j\Omega} - \frac{1}{2}e^{-2j\Omega}}{1 + \frac{1}{6}e^{-j\Omega} - \frac{1}{6}e^{-2j\Omega}}. \]

Finally, to obtain the impulse response, we first decompose the expression for \( H(z) \) into partial fractions:

\[ H(D) = 3 - \frac{\frac{13}{20}}{1 - \frac{1}{3}D} - \frac{\frac{21}{10}}{1 + \frac{1}{2}D}. \]

The impulse response \( h[n] \) is just the coefficient of \( D^n \) in the expansion of \( H(D) \) as a power series in \( D \), and so:

(c) \[ h[n] = 3\delta[n] - \frac{13}{20} \left( \frac{1}{3} \right)^n - \frac{21}{10} \left( -\frac{1}{2} \right)^n, \]

for \( n \geq 0. \)
Problem 5. (Graded by Cedric)

(a) The easy way to do this problem is to observe that “the triangle is the convolution of the box with itself,” i.e.,

\[ x(t) = y(t) \ast y(t), \]

where

\[ y(t) = \begin{cases} 
1 & \text{if } |t| \leq 1/2 \\
0 & \text{if } |t| > 1/2. 
\end{cases} \]

Using the result of Example 4.4 (or Table 4.2), we have

\[ Y(j\omega) = 2 \sin \frac{\omega}{2}, \]

so that

\[ X(j\omega) = Y(j\omega)^2 = 4 \frac{\sin^2 \frac{\omega}{2}}{\omega^2}. \]

(b)

Sketch of \( x(t) \ast \sum_{k=-\infty}^{+\infty} \delta(t - 4k) \).

(c) Here are two such \( g(t) \)'s (there are infinitely many more):