Designing Discrete-Time Filters for Differentiation and Integration

In these notes I will discuss the design and implementation of discrete-time filters for differentiation and integration. Together with the “sample, process in discrete time, and interpolate” paradigm of Chapter 7, these filters will allow us to build systems to differentiate and integrate continuous time signals using digital signal processing.

1. Discrete-Time Differentiation.

As discussed in class (see also OW2, Sec. 7.4.1) the ideal digital differentiator has frequency response

\[ H(e^{j\Omega}) = j\Omega, \text{ for } |\Omega| \leq \pi, \]

and repeats with period 2\pi. We wish to find a class of FIR filters which approximate this ideal as closely as possible. Since the impulse response corresponding to the ideal system (1) is real and odd, it is reasonable to investigate FIR approximations of the same form. Thus we will restrict our attention to filters characterized by system functions of the form

\[ H_N(z) = \sum_{n=1}^{N} a_n (z^n - z^{-n}), \]

and call \( N \) the order of the approximation.

Substituting \( z = e^{j\Omega} \) in (2), we find that the frequency response of the \( N \)th order approximation is

\[ H_N(e^{j\Omega}) = j \sum_{n=1}^{N} a_n \sin(n\Omega). \]

Comparing (1) to (3), we see that the design problem boils down to choosing the filter coefficients \((a_n)_{n=1}^{N}\) so that

\[ f_N(\Omega) = \sum_{n=1}^{N} a_n \sin(n\Omega) \approx \Omega, \]

\[ ^1 \text{In fact, this impulse response is } h[0] = 0, h[n] = (-1)^n / n, \text{ for } n \neq 0. \] (See OW2, Ex. 7.2.)
in some appropriate sense. Although there are several possible criteria by which to interpret "\( \approx \)”, for example minima; mean squared error, we will investigate filters which are optimal in the sense of maximal tangency to the ideal (1) at \( \Omega = 0 \).

If we use the Taylor series for \( \sin x \), viz.,

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots
\]

we can develop \( f_N(\Omega) \) as defined in (4) as a power series in \( \Omega \), viz.,

\[
f_N(\Omega) = S_1\Omega + S_3\Omega^3 + \cdots,
\]

where the \( S_k \)'s are linear combinations of the \( a_n \)'s. Since \( f_N(\Omega) \) has \( N \) undetermined coefficients, it is reasonable to try to choose these coefficients so that

\[
S_1 = 1, \quad S_3 = \cdots = S_{2N-1} = 0,
\]

so that

\[
(5) \quad f_N(\Omega) = \Omega + O(\Omega^{2N+1}).
\]

For example, with \( N = 1 \), we have

\[
f_1(\Omega) = a_1 \sin \Omega = a_1(\Omega - \frac{\Omega^3}{3!} + \cdots),
\]

so that if we choose \( a_1 = 1 \), we get

\[
f_1(\Omega) = \Omega + O(\Omega^3),
\]

which is the closest match to the desired \( f(\Omega) = \Omega \) possible with just one degree of freedom. The system function for the corresponding filter is

\[
H_1(z) = \frac{1}{2}(z - z^{-1}).
\]

In the (discrete) time domain this filter is characterized by the difference equation

\[
y[n] = \frac{1}{2}(x[n+1] - x[n-1]).
\]

which corresponds to the well-known (continuous-time) approximation

\[
x'(t) \approx \frac{x(t+T) - x(t-T)}{2T}.
\]
Similarly, with $N = 2$, we get
\[
f_1(\Omega) = a_1(\Omega - \frac{\Omega^3}{3!} + \cdots) + a_2(2\Omega - \frac{8\Omega^3}{3!} + \cdots)
\]
\[
= (a_1 + 2a_2)\Omega - \frac{(a_1 + 8a_2)}{3!}\Omega^3 + \cdots
\]

Thus if we select $a_1$ and $a_2$ so that
\[
a_1 + 2a_2 = 1
\]
\[
a_1 + 8a_2 = 0,
\]
i.e., $a_1 = 4/3$, $a_2 = -1/6$, we get
\[
f_2(\Omega) = \Omega + O(\Omega^5).
\]

The corresponding system function is
\[
H_2(z) = \frac{2}{3}(z - z^{-1}) - \frac{1}{12}(z^2 - z^{-2}),
\]
with difference equation
\[
y[n] = -\frac{1}{12}x[n + 2] + \frac{2}{3}x[n + 1] - \frac{2}{3}x[n - 1] + \frac{1}{12}x[n + 2].
\]

This corresponds to the not-so-well known approximation
\[
x'(t) \approx \frac{4}{3} \frac{x(t + T) - x(t - T)}{2T} - \frac{1}{3} \frac{x(t + 2T) - x(t - 2T)}{4T}.
\]

Figure 1 shows an implementation of a causal version of this system, viz.,
\[
y[n] = -\frac{1}{12}x[n] + \frac{2}{3}x[n - 1] - \frac{2}{3}x[n - 3] + \frac{1}{12}x[n - 4].
\]

**Figure 1.** A causal FIR filter implementing a $N = 2$ differentiator.
In general, we can expand $f_n(\Omega)$ as a power series in $\Omega$ as follows:

$$f_N(\Omega) = \sum_{n=1}^{N} a_n \sin(n\Omega)$$

$$= \sum_{n=1}^{N} a_n \sum_{m \geq 0} (-1)^m \frac{(n\Omega)^{2m+1}}{(2m+1)!}$$

$$= \sum_{m \geq 0} (-1)^m \frac{\Omega^{2m+1}}{(2m+1)!} \sum_{n=1}^{N} a_n n^{2m+1}$$

Thus to ensure that (5) holds, we choose the filter coefficients $(a_n)$ so that

$$\sum_{n=1}^{N} a_n n^{2m+1} = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m = 1, \ldots, N - 1. \end{cases}$$

This is a set of $N$ linear equations in the $N$ unknowns $a_1, \ldots, a_N$, which are easily solved by computer. Here is a short table of the solutions.

$n = 1 \quad n = 2 \quad n = 3 \quad n = 4 \quad n = 5$

$N = 1 : \quad 2/2$

$N = 2 : \quad 4/3\quad -1/6$

$N = 3 : \quad 6/4\quad -3/10\quad 2/60$

$N = 4 : \quad 8/5\quad -6/15\quad 8/105\quad -1/140$

$N = 5 : \quad 10/6\quad -10/21\quad 20/168\quad -5/252\quad 1/630$

Figure 2 shows a plot of the optimized functions $f_1(\Omega), \ldots, f_5(\Omega)$ vs. $\Omega$.

**Figure 2.** The first five “maximally flat” discrete-time differentiators.
2. Discrete-Time Integration.

Motivated by the trapezoid rule and Simpson’s rule, perhaps, we consider discrete-time integrating filters characterized by difference equations of the form

\[ y[n] - y[n - N] = \sum_{k=0}^{N} a_k x[n - k], \]

where the filter coefficients \(a_k\) satisfy the symmetry condition

\[ a_k = a_{N-k}, \text{ for } k = 0, \ldots, N, \]

and \(N\) is called the order of the filter. The system function corresponding to the difference equation (6) is

\[ H_N(z) = \frac{\sum_{k=0}^{N} a_k z^{-k}}{1 - z^{-N}} = \frac{\sum_{k=0}^{N} a_k z^{N/2-K}}{z^{N/2} - z^{-N/2}}. \]

Our goal is to select the coefficients \(a_k\) so that the corresponding frequency response \(H_N(e^{j\Omega})\) is as close as possible (in a suitable sense) to the ideal frequency response for a digital integrator, viz.,

\[ H(e^{j\Omega}) = \frac{1}{j\Omega} \]

as possible.

For example, with \(N = 1\), the system function in (9) is (recalling that \(a_0 = a_1\) by (7))

\[ H_1(z) = \frac{a_0(z^{1/2} + z^{-1/2})}{z^{1/2} - z^{-1/2}}. \]

The corresponding frequency response is then easily seen to be

\[ H_1(e^{j\Omega}) = \frac{1}{j} \left( \frac{a_0 \cos(\Omega/2)}{\sin(\Omega/2)} \right). \]

Comparing (10) to (11), we see that in this case the problem is to choose the coefficient \(a_0\) so that

\[ \frac{a_0 \cos(\Omega/2)}{\sin(\Omega/2)} \approx \frac{1}{\Omega}, \]

or alternatively

\[ \Omega a_0 \cos(\Omega/2) \approx \sin(\Omega/2). \]
If we expand both sides of (12) as power series in $\Omega$, using the Taylor series
\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \]
\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \]
(12) becomes (ignoring terms of order $\Omega^5$ and higher)
\[ a_0(\Omega - \frac{\Omega^3}{8} + \cdots) \approx \frac{\Omega}{2} - \frac{\Omega^3}{48} + \cdots. \]
Thus if we take as our measure of goodness the agreement of the power series at $\Omega = 0$, the best choice for $a_0$ is $a_0 = 1/2$, in which case the filter’s system function is
\[ H_1(z) = \frac{(1/2)(1 + z^{-1})}{1 - z^{-1}}, \]
corresponding to the difference equation
\[ y[n] = y[n - 1] + \frac{1}{2}(x[n] + x[n - 1]), \]
which, as discussed in class, is the trapezoid rule for integration.

Similarly, for $N = 2$ the system function is by (9)
\[ H_2(z) = \frac{a_1 + a_0(z + z^{-1})}{z - z^{-1}}. \]
Substituting $z = e^{j\Omega}$, this becomes
\[ H_2(e^{j\Omega}) = \frac{1}{j} \left( \frac{a_1/2 + a_0 \cos \Omega}{\sin \Omega} \right). \]
Reasoning as in the $N = 1$ case, we see that the problem is to select the coefficients $a_0$ and $a_1$ so that
\[ \Omega(a_1/2 + a_0 \cos \Omega) \approx \sin \Omega. \]
Expanding both sides of (13) in power series in $\Omega$ up to and including terms of order $\Omega^3$, we get
\[ (a_1/2 + a_0)\Omega - (a_0/2)\Omega^3 + \cdots \approx \Omega - \frac{1}{6}\Omega^3 + \cdots. \]
Thus we can get equality in (13) up to an including terms of order $\Omega^4$ if $a_0$ and $a_1$ are chosen so that
\[ a_0 + \frac{1}{2}a_1 = 1 \]
\[ \frac{1}{2}a_0 = \frac{1}{6}. \]
The solution is $a_0 = 1/3$, $a_1 = 4/3$, corresponding to the system function

$$H_2(z) = \frac{\frac{1}{3} + \frac{4}{3}z^{-1} + \frac{1}{3}z^{-2}}{1 - z^{-2}},$$

with corresponding difference equation

$$y[n] = y[n-2] + \frac{1}{3}x[n] + \frac{4}{3}x[n-1] + \frac{1}{3}x[n-2],$$

which is, as discussed in class, nothing more or less than Simpson’s rule.