

The Hurwitz Stability Criterion

In these notes I will discuss the “Hurwitz Stability criterion,” which I covered in class on March 8 and 11. This provides a practical method of testing whether a causal rational system (in continuous or discrete time) is stable, or not.

1. Continuous time.

Here is the basic problem addressed by the Hurwitz Criterion: Given an n th degree polynomial of the form

$$(1) \quad P(s) = s^n + a_1 s^{n-1} + \cdots + a_n,$$

where the coefficients are real, determine if $P(s)$ is “stable,” i.e. if all complex solutions to the equation $P(s) = 0$ have negative real part.

The first thing to notice is that a *necessary* condition for stability is that all of the coefficients a_i be positive. This is because *any* $P(s)$ of the form (1) can be factored as

$$(2) \quad P(s) = \prod_j (s - \alpha_j) \prod_k (s - \beta_k)(s - \beta_k^*),$$

where the $\{\alpha_j\}$'s are the real zeros, and the $\{\beta_k, \beta_k^*\}$ are the complex conjugate pairs of zeros, of $P(s)$. But if $P(s)$ is stable we know that each α_j is negative, so that $(s - \alpha_j) = (s + A_j)$ with $A_j > 0$, and $(s - \beta_k)(s - \beta_k^*) = s^2 + B_k s + C_k$ with $B_k = -2 \operatorname{Re}(\beta_k) > 0$ and $C_k = |\beta_k|^2 > 0$. Thus (2) becomes

$$(3) \quad P(s) = \prod_j (s + A_j) \prod_k (s^2 + B_k s + C_k),$$

with A_j, B_k, C_k all positive, so that in the expansion of (3), all coefficients are positive.

This simple necessary condition is also sufficient for $n = 1$ and $n = 2$, as is easily checked. But for $n \geq 3$, it no longer suffices, as is illustrated by

$$P(s) = s^3 + s^2 + s + 1 = (s + 1)(s^2 + 1),$$

which has positive coefficients but is unstable. The Hurwitz test, which is explained below, provides a necessary and sufficient condition for stability.

We illustrate it first for $n = 6$, in which case the polynomial $P(s)$ in (1) becomes

$$s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s + a_6.$$

The corresponding *Hurwitz matrix* is

$$H_6 = \begin{pmatrix} a_1 & a_3 & a_5 & & & \\ 1 & a_2 & a_4 & a_6 & & \\ & a_1 & a_3 & a_5 & & \\ & 1 & a_2 & a_4 & a_6 & \\ & & a_1 & a_3 & a_5 & \\ & & 1 & a_2 & a_4 & a_6 \end{pmatrix}$$

(The generalization to arbitrary n is clear if you notice that the main diagonal of H_n is (a_1, a_2, \dots, a_n) .)

Theorem 1. $P(s)$ is stable if and only if the leading principal minors of H_n are all positive. (The leading principal minors are the determinants of the upper left 1×1 , 2×2 , \dots , $n \times n$ submatrices of H_n . In what follows we will denote the upper left $k \times k$ minor by Δ_k .)

For $n = 6$, Theorem 1 says that $P(s)$ is stable if and only if

$$\begin{aligned} \Delta_1 &= a_1 > 0 \\ \Delta_2 &= \det \begin{pmatrix} a_1 & a_3 \\ 1 & a_2 \end{pmatrix} > 0 \\ \Delta_3 &= \det \begin{pmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{pmatrix} > 0 \\ &\vdots \\ \Delta_6 &= \det \begin{pmatrix} a_1 & a_3 & a_5 & & & \\ 1 & a_2 & a_4 & a_6 & & \\ & a_1 & a_3 & a_5 & & \\ & 1 & a_2 & a_4 & a_6 & \\ & & a_1 & a_3 & a_5 & \\ & & 1 & a_2 & a_4 & a_6 \end{pmatrix} > 0 \end{aligned}$$

Example 1. Let $P(s) = s^3 + s^2 + s + 1$. The corresponding Hurwitz matrix is

$$H_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ & 1 & 1 \end{pmatrix}.$$

A simple calculation (almost by inspection) gives $\Delta_1 = 1$, $\Delta_2 = \Delta_3 = 0$, which means $P(s)$ is unstable, as we noted above. ■

Example 2. Let $P(s) = s^4 + 7s^3 + 17s^2 + 17s + 6$. Here the Hurwitz matrix is

$$H_4 = \begin{pmatrix} 7 & 17 \\ 1 & 17 & 6 \\ & 7 & 17 \\ & 1 & 17 & 6 \end{pmatrix}.$$

Here $\Delta_1 = 7$, $\Delta_2 = 102$, $\Delta_3 = 1440$, $\Delta_4 = 8640$, which means $P(s)$ is stable, even though we have no clue as to the location of the roots (apart from the fact that they all have negative real part), ■

Example 3. Let $P(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$. The Hurwitz matrix is

$$H_5 = \begin{pmatrix} 2 & 4 & 10 & & & \\ 1 & 2 & 11 & & & \\ & 2 & 4 & 10 & & \\ & 1 & 2 & 11 & & \\ & & 2 & 4 & 10 & \end{pmatrix}.$$

Here $\Delta_1 = 2$, $\Delta_2 = 0$, stop! We know already that $P(s)$ is unstable. ■

2. Discrete-Time.

In discrete time, the problem is this. Given an n th degree polynomial

$$(4) \quad Q(z) = z^n + b_1 z^{n-1} + \cdots + b_n,$$

determine whether or not all solutions to $Q(z) = 0$ lie inside the unit circle $\{|z| < 1\}$. Here we use a trick, or to use its formal name, the *bilinear transformation*.

$$f(z) = \frac{z-1}{z+1}.$$

It is easy to see that $f(z)$ has the following nice properties:

$$\begin{aligned} \{|z| = 1\} &\rightarrow \{\operatorname{Re}(s) = 0\} \\ f : \{|z| < 1\} &\rightarrow \{\operatorname{Re}(s) < 0\} \\ \{|z| > 1\} &\rightarrow \{\operatorname{Re}(s) > 0\}. \end{aligned}$$

as well as a nice inverse:

$$g(s) = f^{-1}(s) = \frac{1+s}{1-s}.$$

What this means is that if we substitute $(1+s)/(1-s)$ for z in (4), we will “move” all the zeros of Q which are inside the unit circle to the left half plane, those outside the unit circle to the right half plane, and those on the unit circle to the imaginary axis. Making this substitution, we find

$$\begin{aligned} Q\left(\frac{1+s}{1-s}\right) &= \sum_{j=0}^n b_j \left(\frac{1+s}{1-s}\right)^{n-j} \\ &= \frac{1}{(1-s)^n} \sum_{j=0}^n b_j (1+s)^{n-j} (1-s)^j \\ &= \frac{1}{(1-s)^n} P(s), \end{aligned}$$

where $P(s)$ is defined by

$$(5) \quad P(s) = \sum_{j=0}^n b_j (1+s)^{n-j} (1-s)^j.$$

We thus have proved

Theorem 2. *The polynomial $Q(z)$ defined in (4) is stable in discrete time if and only if the transformed polynomial $P(s)$ defined in (5) is stable in continuous time.*

Thus testing for z -stability is almost as simple as testing for s -stability. The only extra step needed is the conversion of the vector of z -coefficients (b_1, \dots, b_n) to the vector of s -coefficients (a_1, \dots, a_n) .

Example 4. Let $Q(z) = (z - 1/2)(z - j/2)(z + j/2) = z^3 - \frac{1}{2}z^2 + \frac{1}{4}z + \frac{1}{8}$, which is of course stable. Applying the transformation in (5), we find

$$\begin{aligned} P(s) &= (1+s)^3 - \frac{1}{2}(1+s)^2(1-s) + \frac{1}{4}(1+s)(1-s)^2 - \frac{1}{8}(1-s)^3 \\ &= \frac{1}{8}(15s^3 + 23s^2 + 21s + 5). \end{aligned}$$

The Hurwitz matrix for $P(s)$ is

$$H_3 = \begin{pmatrix} 23 & 5 & \\ 15 & 21 & \\ & 23 & 5 \end{pmatrix},$$

with leading principal minors $\Delta_1 = 23$, $\Delta_2 = 408$, $\Delta_3 = 2040$. Therefore $P(s)$ and $Q(z)$ are both stable. ■

Example 5. Let $Q(z) = z^3 - 2z^2 + \frac{1}{4}z - \frac{1}{2}$. In this case we have

$$\begin{aligned} P(s) &= (1+s)^3 - 2(1+s)^2(1-s) + \frac{1}{4}(1+s)(1-s)^2 - \frac{1}{2}(1-s)^3 \\ &= \frac{1}{4}(15s^3 + 13s^2 + 9s - 5). \end{aligned}$$

Since $P(s)$ has a negative coefficient, we immediately conclude that $P(s)$, and so also $Q(z)$, is unstable. (Indeed, $Q(z) = (z - 2)(z^2 + 1/4)$.) ■