Excerpt from : The Theory of Functions
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CHAPTER VII

POWER SERIES WITH A FINITE RADIUS OF CONVERGENCE

7.1. The circle of convergence. We know that every power series has a circle of convergence, within which it converges, and outside which it diverges. The radius of this circle may, however, be infinite, so that the circle includes the whole plane. In this chapter we shall consider power series which have a finite radius of convergence.

The radius of convergence of a power series is determined by the moduli of the coefficients in the series.

The power series

$$\sum_{n=0}^{\infty} a_n z^n \tag{1}$$

has the radius of convergence

$$R = \varliminf_{n\to\infty} |a_n|^{-1/n}. \tag{2}$$

Suppose that R is defined by (2). If z is a point where the series (1) converges, $a_n z^n \to 0$ as $n \to \infty$. Hence, if n is sufficiently large,

$$|a_n z^n| < 1,$$

i.e.

$$|z| < |a_n|^{-1/n}.$$

Making $n \to \infty$, it follows that $|z| \leqslant R$. Hence the radius of convergence does not exceed R.

On the other hand, for sufficiently large values of n,

$$|a_n|^{-1/n} > R-\epsilon,$$

i.e.

$$|a_n| < (R-\epsilon)^{-n}.$$

Hence the series (1) is convergent if $\sum (R-\epsilon)^{-n}|z|^n$ is convergent, i.e. if $|z| < R-\epsilon$. Since ϵ is arbitrarily small, the series (1) is convergent if $|z| < R$. Thus the radius of convergence is at least equal to R. Putting together the two results, the theorem follows.

Examples. (i) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{2n!}{(n!)^2} z^n, \quad \sum_{n=1}^{\infty} \frac{n!}{n^n} z^n, \quad \sum_{n=0}^{\infty} n!\, z^{n^2}.$$

(ii) If $R = 1$, and the only singularities on the unit circle are simple poles, then a_n is bounded. [For

$$f(z) = \frac{a}{1-ze^{-i\alpha}} + \frac{b}{1-ze^{-i\beta}} + \dots + \frac{k}{1-ze^{-i\kappa}} + g(z),$$

where $g(z)$ is regular for $|z| < 1+\delta$ $(\delta > 0)$. Hence $g(z) = \sum b_n z^n$, where $b_n = o(1)$.]

(iii) If $R = 1$, and the only singularities on the unit circle are poles of order p, then $a_n = O(n^{p-1})$.

7.11. We also know from the Cauchy-Taylor theorem that the circle of convergence of the series passes through the singularity or singularities of the function which are nearest to the origin. *Hence the modulus of the nearest singularity can be determined from the moduli of the coefficients in the series.*

7.2. Position of the singularities. While the modulus of the nearest singularities is determined in quite a simple way, their exact position is not usually so easy to find. There are, however, some special cases in which we can identify a particular point as a singularity.

In the following theorems we shall take the radius of convergence to be unity; we can, of course, pass from this to the general case by a simple transformation.

7.21. *If $a_n \geqslant 0$ for all values of n, then $z = 1$ is a singular point.*

Suppose, on the contrary, that $z = 1$ is regular. Then, if we take a point ρ on the real axis between 0 and 1, there is a circle with centre ρ which includes the point 1, and in which the function is regular. If $f(z)$ is the function, the Taylor's series about ρ is

$$\sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(\rho)}{\nu!} (z-\rho)^\nu, \tag{1}$$

and this converges at a point $z = 1+\delta$ $(\delta > 0)$. Now

$$f^{(\nu)}(\rho) = \sum_{n=\nu}^{\infty} n(n-1)\dots(n-\nu+1)a_n \rho^{n-\nu}, \tag{2}$$

and so the above series is

$$\sum_{\nu=0}^{\infty} \frac{(z-\rho)^\nu}{\nu!} \sum_{n=\nu}^{\infty} n(n-1)\dots(n-\nu+1)a_n \rho^{n-\nu}.$$

This is a double series of positive terms, convergent for $z = 1+\delta$.