Some Convergence Theory for \( z \)-Transforms

In this handout I will present (with modest justification) formulas which describe the regions of convergence for \( z \)-transforms. This material closely parallels, but is not identical to, the material in OWY, Section 10.2 (for the \( z \)-transform).

1. The \( Z \)-Transform.

If \( x[n] \) is a discrete-time signal, its \( z \)-transform is the function \( X(z) \) of the complex variable \( z \) defined by a two-sided power series (which is technically called a Laurent series), as follows.

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.
\]

We wish to know when the sum in (1) converges, and here is the main result.

**Theorem 1.** Define nonnegative real numbers \( r_R \) and \( r_L \) as follows:

\[
\begin{align*}
    r_R &= \lim \sup_{n \to \infty} |x[n]|^{1/n} \\
r_L &= \lim \inf_{n \to \infty} |x[-n]|^{-1/n} \\
    &= \lim \inf_{n \to -\infty} |x[n]|^{1/n}.
\end{align*}
\]

If \( r_R < |z| < r_L \), then \( X(z) \) converges absolutely. On the other hand if \( |z| > r_L \) or if \( |z| < r_R \), then \( X(z) \) diverges.

**Proof:** We break \( X(z) \) into two parts, the right part \( X_R(z) \) and the left part \( X_L(z) \):

\[
\begin{align*}
    X_R(z) &= \sum_{n \geq 0} x[n]z^{-n} \\
    X_L(z) &= \sum_{n \leq -1} x[n]z^{-n} = \sum_{n \geq 1} x[-n]z^n.
\end{align*}
\]

Then \( X(z) = X_R(z) + X_L(z) \), and by definition, \( X(z) \) converges if and only if both \( X_R(z) \) and \( X_L(z) \) converge. To determine when these two one-sided series converge, we invoke the following famous result from complex variables.

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* The symbols “\( \lim \sup \)” and “\( \lim \inf \)” appearing in (2), (3), and elsewhere in this writeup are the so-called “limit superior” and “limit inferior,” or upper and lower limits, whose formal definitions can be found, e.g. in Apostol’s *Mathematical Analysis*, p. 184. Taking the ordinary limit instead will always give the same answer, if the limit exists. This more general formulation will always work, even if the limit doesn’t exist.
Theorem 2. A power series of the form

\[ f(z) = \sum_{n \geq 0} a_n z^n \]

(where the \( a_n \)'s are complex numbers) converges absolutely for \( |z| < r \) and diverges for \( |z| > r \), where the nonnegative number \( r \) (the radius of convergence) is given by the formula

\[ r = \lim \inf_{n \to \infty} |a_n|^{-1/n}. \]

Furthermore, the function \( f(z) \) must have one or more singularities on the circle \( \{ z : |z| = r \} \).

If we take Theorem 2 for granted, Theorem 1 follows quite easily. The right part \( X_R(z) \) defined in (5) is a power series in \( z^{-1} \), and so by Theorem 2 it converges absolutely if \( |z^{-1}| < s \), and diverges for \( |z^{-1}| > s \), where \( s \) is defined by the formula

\[ s = \lim \inf_{n \to \infty} |x[n]|^{-1/n}. \]

To make this a condition on \( |z| \) instead of \( z^{-1} \), we just invert the limit in (9). The result is that \( X_r(z) \) converges for \( |z| > r_R \) and diverges for \( |z| < r_R \), where \( r_R \) is as defined in (2). Similarly, if we apply Theorem 2 to the left part of \( X(z) \) as defined in (6), we see that \( X_L(z) \) converges for \( |z| < t \) and diverges for \( |z| > t \), where \( t \) is given by

\[ t = \lim \inf_{n \to \infty} |x[-n]|^{-1/n} = \lim \inf_{n \to -\infty} |x[n]|^{1/n} = r_L, \]

where \( r_L \) is as defined in (3) or (4). Putting these two parts together, we get the result stated in Theorem 1.

The result of Theorem 1 is summarized by saying that \( \{ z : r_R < |z| < r_L \} \) is the region of convergence * for the \( z \)-transform \( X(z) \). However, this is somewhat of a misnomer, since \( X(z) \) may also converge on part or all of the critical circles \( C_L = \{ z : |z| = r_L \} \) or \( C_R = \{ z : |z| = r_R \} \). Thus, for future reference, we define the open ROC and the closed ROC as follows.

\[ \text{open ROC} = \{ z : r_R < |z| < r_L \} \]
\[ \text{closed ROC} = \{ z : r_R \leq |z| \leq r_L \} \]

* In OW2 the ROC is defined to be the set of \( z \)'s for which the series (1) converges absolutely. This is a small, but sometimes important, difference. We have chosen our definition because with it, the ROC can be calculated with the help of Theorem 1.
Normally, however, when we say the ROC, we will mean the open ROC. The following examples may clear things up.

**Example 1.** Let the sequence $x[n]$ be defined as follows:

$$
x[n] = \begin{cases} 
(1/2)^n & \text{if } n \geq 0 \text{ and } n \text{ is even} \\
(1/3)^n & \text{if } n \geq 0 \text{ and } n \text{ is odd} \\
2^n & \text{if } n < 0.
\end{cases}
$$

For $n \geq 0$ we have $|x[n]|^{1/n} = 1/2$ if $n$ is even, and $= 1/3$ if $n$ is odd. Thus from (2) we find that $r_R = 1/2$. For $n < 0$ we have $|x[n]|^{1/n} = 2$, so that by (3) we have $r_L = 2$. Thus the ROC for this example is the region $\{1/2 < |z| < 2\}$. ■

**Example 2.** Now let

$$
x[n] = \begin{cases} 
1 & \text{if } n \geq 0 \\
0 & \text{if } n \leq 0.
\end{cases}
$$

Then from (2), we get $r_R = 1$, and from (3) we get $r_L = \infty$. (Indeed, for any “right-sided” signal $x[n]$, i.e., one for which $x[n] = 0$ for all $n \leq n_0$, we have $r_L = \infty$, and the ROC is $\{|z| > r_R\}$. Similarly, $r_R = 0$ for any left-sided signal, and the ROC is $\{|z| < r_L\}$.) Thus the ROC for this signal is $\{z : |z| > 1\}$. Note that in this case $X(z)$ converges nowhere on the critical circle $\{|z| = 1\}$, since the terms of the series (1) do not approach zero for $|z| = 1$. ■

**Example 3.** Consider next

$$
x[n] = \begin{cases} 
1/n & \text{if } n \geq 1 \\
0 & \text{if } n \leq 0.
\end{cases}
$$

Once again from Theorem 1, we get $r_R = 1$, and $r_L = \infty$. Thus the ROC for this signal is also $\{z : |z| > 1\}$. In this case $X(z)$ diverges for $z = 1$, and (I think) converges for all other values of $z$ on the critical circle $|z| = 1$. ■

**Example 4.** Finally, consider

$$
x[n] = \begin{cases} 
1/n^2 & \text{if } n \geq 1 \\
0 & \text{if } n \leq 0.
\end{cases}
$$

Once again Theorem 1 tells us that $r_R = 1$, and $r_L = \infty$. The interesting thing here is that $X(z)$ converges absolutely everywhere on the critical circle $|z| = 1$, since the series $\sum_{n \geq 1} n^{-2}$ converges. ■

It is normally possible to determine whether or not a discrete-time LTI system is stable by knowing the ROC. For according to OW2, Section 2.3.7, a system with impulse response $h[n]$ is stable if and only if $h[n]$ is absolutely summable, i.e., if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty. \tag{11}$$

The condition (11) is equivalent to saying that the $z$-transform $H(z)$ of $h[n]$ is absolutely convergent on the unit circle. Since we know from Theorem 1 that $H(z)$ is absolutely convergent in the open ROC and diverges outside the closed ROC, we have

**Corollary 1.** If the open ROC for $H(z)$ contains the point $z = 1$, then the system is stable. If the point $z = 1$ lies outside the closed ROC, then the system is unstable. On the other hand, if $z = 1$ lies on one of the critical circles, no general conclusion is possible. ■

We illustrate Corollary 1 by revisiting the four examples from the last section, assuming now that each of the signals considered is the impulse response for an LTI system. In Example 1, the open ROC is $\{z : 1/2 < |z| < 2\}$, which safely contains the unit circle $\{z : |z| = 1\}$, so the system is stable. However, in each of Examples 2–4, the point $z = 1$ lies on the critical circle, so Corollary 1 is no help. In Example 2, we observed that the series for $X(z)$ diverges everywhere on the unit circle, so the system is unstable. In Example 3, we we observed that the series for $X(z)$ diverges for $z = 1$, so again the system is unstable. Finally, in Example 4, we observed that the series for $X(z)$ converges absolutely on the unit circle, so this system is stable!

The forgoing examples illustrate the perils of trying to determine system stability when $z = 1$ lies on one of the critical circles. However, there is one important special case when the problem is easily solved, viz., when $X(z)$ is a rational function. In this case, the only possible singularities of $X(z)$ are poles, and so by Theorem 2, there must be one or more poles of $X(z)$ on each of the critical circles. Naturally, $X(z)$ diverges if $z = z_p$ is a pole of $X(z)$, and so $X(z)$ does not converge absolutely anywhere on the critical circles. Thus we have the following two corollaries.

**Corollary 2.** Suppose $H(z)$ is rational. Then the corresponding system is stable if and only if the unit circle $z = 1$ is contained in the open ROC. ■

**Corollary 3.** Suppose $H(z)$ is rational and causal. Then the corresponding system is stable if and only if $H(z)$ has no poles in the region $|z| \geq 1$. ■