Solutions to Midterm Examination

Problem 1. (The simplest way to do this problem is in the time domain.)

(a) By the sampling theorem, if $x(t) \in C(\omega_s)$, then

$$x(t) = \sum_n x(nT_s) \text{sinc}(\frac{t}{T_s} - n).$$

Differentiating this, we find

$$x'(t) = \sum_n x(nT_s) \frac{1}{T_s} \text{sinc}'(\frac{t}{T_s} - n).$$

Thus one possibility for $h(t)$ is

$$h(t) = \frac{1}{T_s} \text{sinc}'(t)$$
$$= \frac{1}{T_s} \left( \frac{\cos \pi t}{t} - \frac{\sin \pi t}{\pi t^2} \right).$$

A plot of $\text{sinc}'(t)$, which is sometimes called $\text{cosinc}(t)$, is given below.

(b) Yes, the function $h(t)$ is unique, and here’s why. If

$$x'(t) = \sum_n x(nT_s) h(\frac{t}{T_s} - n)$$
for all \( x(t) \in C(\omega_s) \), then substituting \( x(t) = \text{sinc}(t/T_s) \) (legal, since \( \text{sinc}(t/T_s) \in C(\omega_s) \)). Do you see why?), we must have

\[
\frac{d}{dt} \text{sinc}(t/T_s) = \frac{1}{T_s} \cos(t/T_s) = \sum_n \text{sinc}(n) h(t/T_s - n) = h(t/T_s),
\]

which means

\[
h(t) = \frac{1}{T_s} \cos(t).
\]

**Problem 2.** The answer is no, and here’s why. Look at the \( z \)-transform of the signal:

\[
X(z) = \sum_{n=-n_0}^{n_0} x[n]z^{-n} = z^{n_0}(x[-n_0] + x[-n_0 + 1]z^{-1} + \cdots + x[n_0]z^{-2n_0}),
\]

i.e., \( z^{n_0} \) times a polynomial of degree \( \leq 2n_0 \) in \( z^{-1} \). Thus \( X(z) = 0 \) can have at most \( 2n_0 + 1 \) solutions, which precludes \( X(e^{j\Omega}) = 0 \) for all \( \Omega_0 \leq \Omega \leq \pi \), unless \( \Omega_0 = \pi \).

**Problem 3.** This one is like Problems 2–3 in HW4. Assuming \( a_k = a_{-k} \) (reasonable, since \( \Omega^2 \) is an even function), we find that in general

\[
H_N(\Omega) = a_0 + \sum_{k=1}^{N} 2a_k \cos(k\Omega).
\]

We then choose the coefficients \( (a_k) \) so that, when expanded as a power series in \( \Omega^2 \), \( H_N(e^{j\Omega}) \) “matches” \( -\Omega^2 \) up to terms of order \( \Omega^{2N} \).

(a) For \( N = 1 \), we have

\[
H_1(e^{j\Omega}) = a_0 + 2a_1 \cos \Omega \\
= a_0 + 2a_1(1 - \frac{1}{2}\Omega^2 + \cdots) \\
= (a_0 + 2a_1) - a_1\Omega^2 + \cdots \\
= -\Omega^2 + \cdots,
\]

which leads to the equations \( a_0 + 2a_1 = 0, a_1 = 1 \). Thus \( a_0 = -2 \) and \( H_1(z) = z - 2 + z^{-1}, \)

i.e.,

\[
H_1(e^{j\Omega}) = 2 \cos \Omega - 2.
\]

Incidentally, this corresponds to the following continuous-time approximation to the second derivative:

\[
f''(x) \approx \frac{f(x + h) - 2f(x) + f(x - h)}{h^2}.
\]
(b) Similarly (details omitted) $H_2(z) = -\frac{1}{12} z^2 + \frac{4}{3} z - \frac{5}{2} - 2 + \frac{4}{3} z^{-1} - \frac{1}{12} z^{-2}$, and so

$$H_2(e^{j\Omega}) = -\frac{1}{6} \cos 2\Omega + \frac{8}{3} \cos \Omega - \frac{5}{2}.$$  

Below we have plotted $H_1(e^{j\Omega})$ and $H_2(e^{j\Omega})$ vs. $-\Omega^2$, for $-\pi < \Omega < \pi$.

**Problem 4.** (To simplify the notation, let’s assume $T_s = 1$).

(a) $x(t) = \cos(\pi t)$ is the only such signal.

(b) For any $0 \leq \theta < \pi$,

$$x(t, \theta) = \frac{\cos(\pi t + \theta)}{\cos(\theta)} = \cos(\pi t) - \tan(\theta) \sin(\pi t)$$

works. (And clearly the maximum value of $x(t, \theta)$ is $|\sec \theta| \geq 1$.) Below we have plotted $x(t, 0)$ (solid) and $x(t, \pi/4) = \cos(\pi t) - \sin(\pi t))$ (dashed).