EE160 Handout #8
February 14, 2007

The Power Spectral Density of FM/PM Signals

Most communications books (Haykin included) do not study the power spectral density of FM/PM signals and do a lousy job of analyzing their spectra. Nonetheless, it turns out that, if we assume that the modulated signal is Gaussian, finding the power spectral density is not too difficult. This is what I will do below.

Let's start off with a PM signal

$$s(t) = A_c \cos(2\pi f_c t + \theta + k_p m(t)),$$

where $m(\cdot)$ is a zero-mean stationary Gaussian random process with autocorrelation function $R_m(\tau)$ and θ is a random phase uniformly distributed between 0 and 2π.

Due to the uniform distribution on θ it follows that

$$Es(t) = 0.$$

Let's now attempt to find the autocorrelation function of $s(\cdot)$.

$$\begin{aligned}
Es(t)s(t+\tau) &= EA_c^2 \cos(2\pi f_c t + \theta + k_p m(t)) \cos(2\pi f_c (t+\tau) + \theta + k_p m(t+\tau)) \\
&= \frac{A_c^2}{2} [E \cos(4\pi f_c t + 2\pi f_c \tau + k_p(m(t) + m(t+\tau)) + 2\theta) + \\
&\qquad E \cos(2\pi f_c \tau + k_p(m(t+\tau) - m(t)))] \\
&= E\frac{A_c^2}{2} \cos(2\pi f_c \tau + k_p(m(t+\tau) - m(t))) \\
&= E\frac{A_c^2}{2} \text{Real}\left[e^{j2\pi f_c \tau + jk_p(m(t+\tau)-m(t))}\right] \\
&= \frac{A_c^2}{2} \text{Real}\left[e^{j2\pi f_c \tau} E e^{jk_p(m(t+\tau)-m(t))}\right].
\end{aligned}$$

Now $m(t)$ and $m(t+\tau)$ are jointly Gaussian with density

$$p_{M(t),M(t+\tau)}(m_1, m_2) = \frac{1}{2\pi\sqrt{\det R}} \exp\left(-\frac{1}{2} \begin{bmatrix} m_1 & m_2 \end{bmatrix} R^{-1} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}\right).$$

where

$$R = \begin{bmatrix} R_m(0) & R_m(\tau) \\ R_m(\tau) & R_x(0) \end{bmatrix}.$$

Therefore we may write

$$Ee^{jk_p(m(t+\tau)-m(t))} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dm_1 dm_2}{2\pi\sqrt{\det R}} \exp\left(-\frac{1}{2} \begin{bmatrix} m_1 & m_2 \end{bmatrix} R^{-1} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} + jk_p(m_2 - m_1)\right).$$

A completion-of-squares argument shows that the exponent in the above integral can be rewritten as

$$-\frac{1}{2}\left(\left[\begin{array}{c} m_1 \\ m_2 \end{array}\right] + R\left[\begin{array}{c} jk_p \\ -jk_p \end{array}\right]\right)^T R^{-1}\left(\left[\begin{array}{c} m_1 \\ m_2 \end{array}\right] + R\left[\begin{array}{c} jk_p \\ -jk_p \end{array}\right]\right) + k_p^2\left(R_m(\tau) - R_m(0)\right).$$

(Check this!) The integral therefore becomes

$$e^{k_p^2(R_m(\tau)-R_m(0))}\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dm_1 dm_2}{2\pi\sqrt{\det R}}\exp\left\{-\frac{1}{2}\left(\left[\begin{array}{c} m_1 \\ m_2 \end{array}\right] + R\left[\begin{array}{c} jk_p \\ -jk_p \end{array}\right]\right)^T R^{-1}\left(\left[\begin{array}{c} m_1 \\ m_2 \end{array}\right] + R\left[\begin{array}{c} jk_p \\ -jk_p \end{array}\right]\right)\right\}}_{=1},$$

which is a simple Gaussian integral. Therefore $Ee^{jk_p(m(t+\tau)-m(t))} = e^{k_p^2(R_m(\tau)-R_m(0))}$ and so

$$Es(t)s(t+\tau) = \frac{A_c^2}{2}\text{Real}\left[e^{j2\pi f_c\tau}e^{k_p^2(R_m(\tau)-R_m(0))}\right].$$

Therefore

$$R_s(\tau) = \frac{A_c^2}{2}e^{-k_p^2 R_m(0)+k_p^2 R_m(\tau)}\cos(2\pi f_c\tau), \tag{1}$$

which is the desired autocorrelation function of the PM signal.

To compute the power spectral density of $s(\cdot)$, let us use the expansion

$$R_s(\tau) = \frac{A_c^2}{2}e^{-k_p^2 R_m(0)}\left[1 + k_p^2 R_m(\tau) + \frac{1}{2}k_p^4 R_m^2(\tau) + \ldots\right]\cos(2\pi f_c\tau).$$

Therefore

$$S_s(f) = \frac{A_c^2}{2}e^{-k_p^2 R_m(0)}\left[\delta(f) + k_p^2 S_m(f) + \frac{1}{2}k_p^4 S_m(f)\star S_m(f) + \ldots\right]\star\left[\frac{1}{2}\delta(f-f_c) + \frac{1}{2}\delta(f-f_c)\right]. \tag{2}$$

Note the increase in the bandwidth of the baseband signal

$$S_{s_b}(f) = \delta(f) + k_p^2 S_m(f) + \frac{1}{2!}k_p^4 S_m(f)\star S_m(f) + \frac{1}{3!}k_p^6 S_m(f)\star S_m(f)\star S_m(f) + \ldots$$

due to all the convolutions.

FM Signals

For FM signals we note that $m(t) = \int_{-\infty}^{t} g(t)dt$, where now $g(\cdot)$ is the message. Since

$$S_m(f) = \frac{1}{4\pi f^2}S_g(f),$$

all we need to do is replace $S_m(f)$ in (2) with the above expression.