2.43 1. (a) Consider the function $w(\theta) = \mathrm{sgn}[\cos(\theta)]$. This function is periodic in θ with period 2π. So It can be expressed in the form of a Fourier series. Since it is an even function, its Fourier series has only cosine terms.Hence

$$w(\theta) = \sum_{n=0}^{\infty} a_n \cos(n\theta) \tag{1}$$

Where

$$a_0 = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{sgn}[\cos(\theta)]d\theta = 0$$

And for non-zero n we have

$$a_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{sgn}[\cos(\theta)]\cos(n\theta)d\theta = 2(1-(-1)^n)\frac{\sin(n\frac{\pi}{2})}{\pi n} \tag{2}$$

Now from (2), we see that a_n is zero for even numbers and thus

$$a_{2n+1} = \frac{4(-1)^n}{\pi(2n+1)}$$

Substituting this in (1) we have

$$\mathrm{w}(\theta) = \frac{4}{\pi}\sum_{\mathrm{n}=0}^{\infty}\frac{(-1)^{\mathrm{n}}}{2\mathrm{n}+1}\cos((2\mathrm{n}+1)\theta) \quad (3)$$

Now since $a(t)$ is positive as given in Problem 2.42 we have

$$z(t) = \mathrm{sgn}[\cos(\underbrace{2\pi f_c t + \phi(t)}_{\theta(\mathrm{t})})] = \mathrm{w}(\theta(\mathrm{t}))$$

So by using (3) we have

$$z(t) = \mathrm{w}(\theta(\mathrm{t})) = \frac{4}{\pi}\sum_{\mathrm{n}=0}^{\infty}\frac{(-1)^{\mathrm{n}}}{2\mathrm{n}+1}\cos(2\pi\mathrm{f_c}(2\mathrm{n}+1) + (2\mathrm{n}+1)\phi(\mathrm{t})) \quad (4)$$

(b) Since the bandwidth B_T of band-pass filter is much less than f_c, so the filter will keep only the first harmony of f_c and cancel all the other harmonics in (4). Therefore the output of the filter is

$$y(t) = \frac{4}{\pi}\cos[2\pi f_c t + \phi(t)]$$

By comparing this output with the original modulated signal we see that by passing the modulated wave through a hard limiter and a band-pass filter we will get a constant amplitude signal without losing original data.

Problem 2.4. (a) Define $g(t) = \int_{-\infty}^{t} m(t)dt$. Then we have

$$S_g(f) = \frac{1}{4\pi^2 f^2}S_m(f) = \frac{1}{4\pi^2 f^2}\sqrt{\frac{8\pi}{W}}f^2 e^{-\frac{f^2}{2W}}$$

So

$$S_g(f) = \frac{1}{\pi\sqrt{2\pi W}}\exp\frac{-f^2}{2W} \quad (5)$$

By taking the inverse Fourier transform of (5) and using the method of Completion of Square we have

$$R_g(\tau) = \int_{-\infty}^{\infty}\frac{1}{\pi\sqrt{2\pi W}}e^{-\frac{f^2}{2W}}e^{j2\pi f\tau}df = \frac{e^{-2W\pi^2\tau^2}}{\pi}\underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi W}}e^{-\frac{(f-j2\pi\tau W)^2}{2W}}df}_{=1}$$

Where the integral is just a Gaussian integral. so

$$R_g(\tau) = \frac{e^{-2W\pi^2\tau^2}}{\pi}$$

and

$$e^{-2W\pi^2\tau^2} \rightleftharpoons \frac{1}{\sqrt{2\pi W}}\exp\frac{-f^2}{2W} \quad (6)$$

Now using (1) in Handout 10 and substituting k_p with $2\pi k_f$ we have

$$R_s(\tau) = \frac{1}{2} e^{-(2\pi k_f)^2 + (2\pi k_f)^2 e^{-2W\pi^2\tau^2}} \cos(2\pi f_c \tau) \tag{7}$$

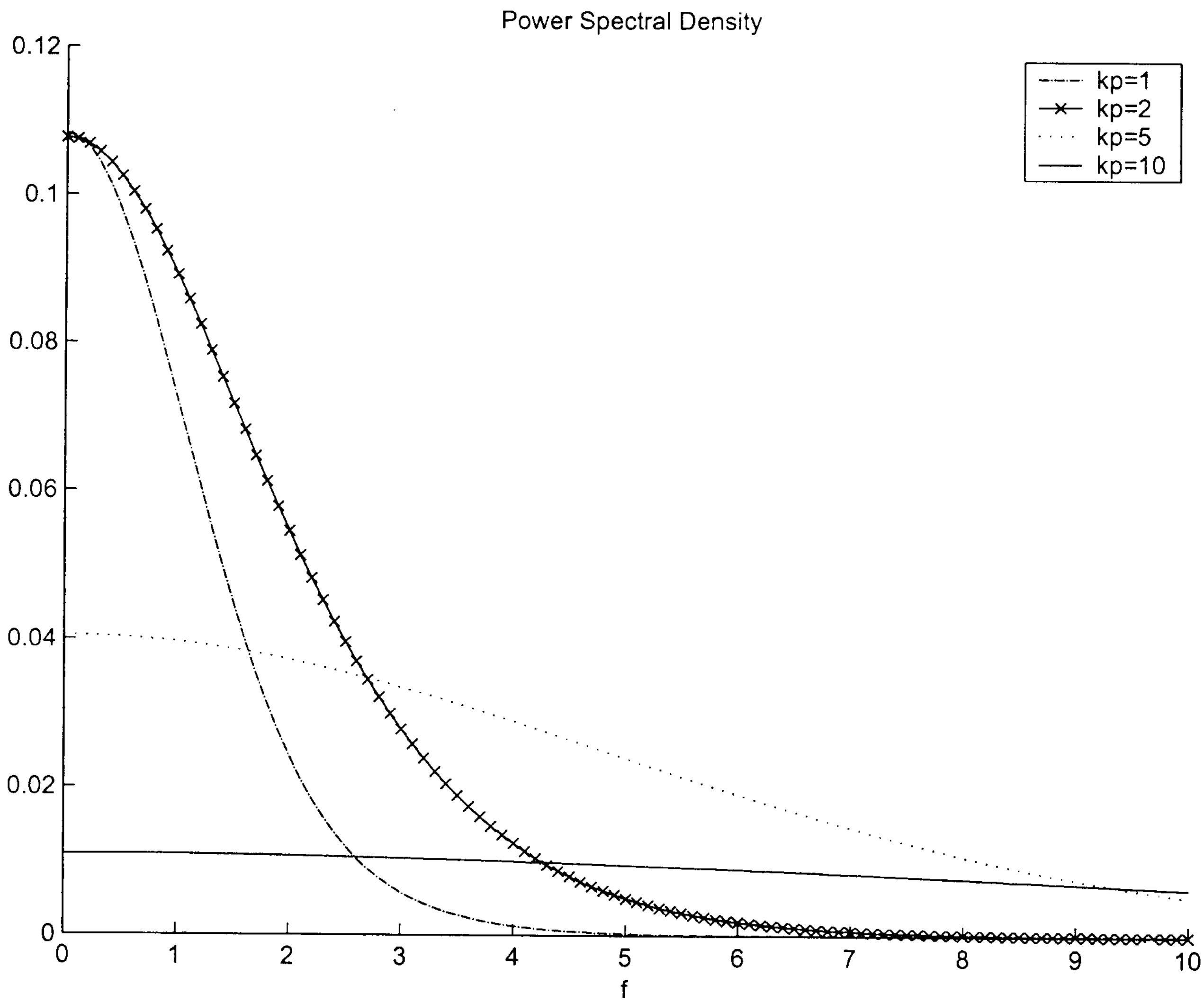
By using the Taylor expansion, (7) can be written as

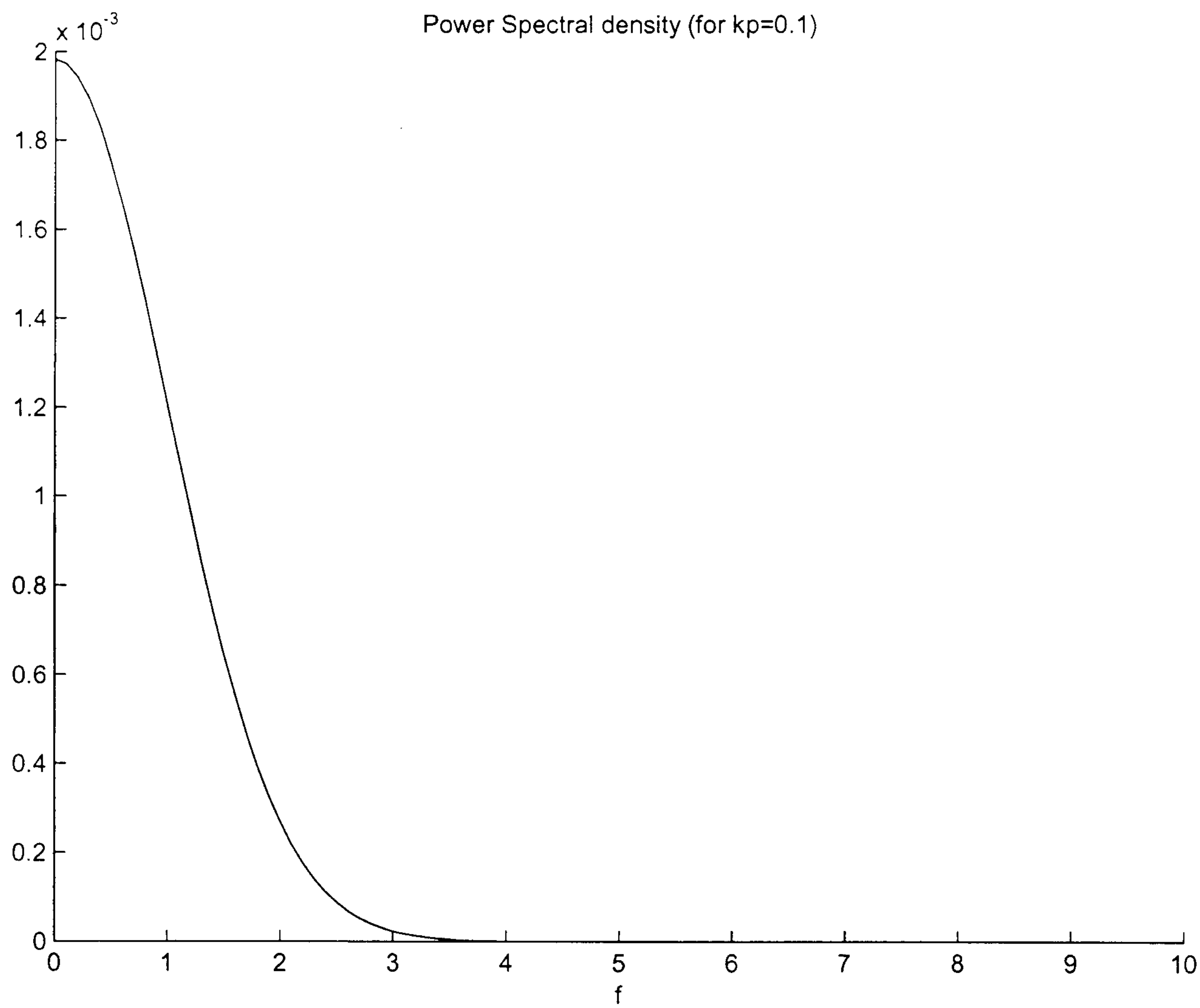
$$R_s(\tau) = \frac{1}{2} e^{-(2\pi k_f)^2} \left[\sum_{n=0}^{\infty} \frac{1}{n!} (2\pi k_f)^{2n} e^{-2nW\pi^2\tau^2} \right] \cos(2\pi f_c \tau)$$

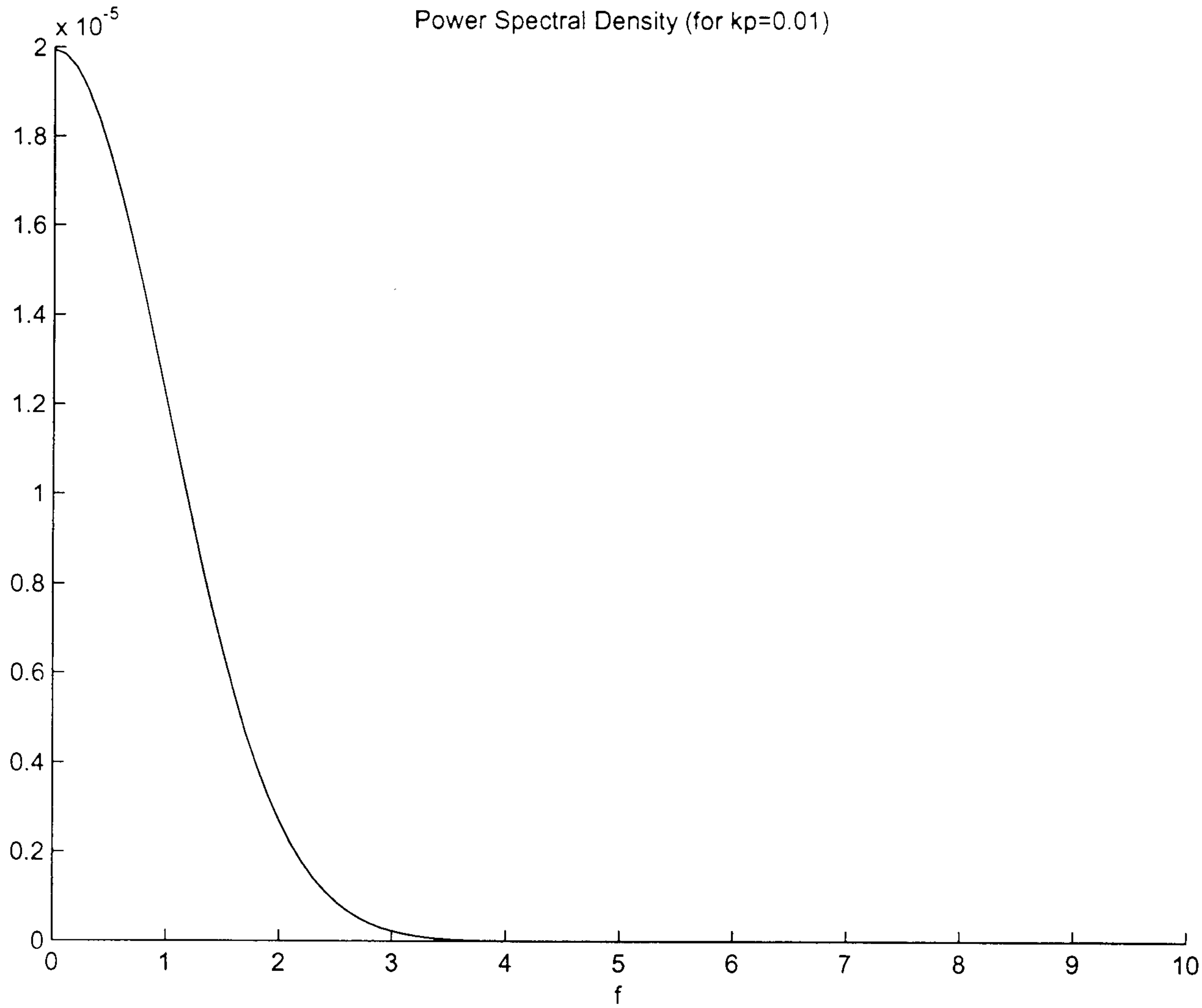
Therefore by using (6) for each term in the above summation we have

$$S_s(f) = \frac{1}{2} e^{-(2\pi k_f)^2} \left[\delta(f) + \sum_{n=1}^{\infty} \frac{1}{n!\sqrt{2\pi n W}} (2\pi k_f)^{2n} e^{-\frac{f^2}{2nW}} \right] * \left[\frac{1}{2}\delta(f - f_c) + \frac{1}{2}\delta(f + f_c) \right]$$

(b) Buy looking at the plots we see that as we increase $2\pi k_f$ the bandwidth of the resulting FM signal increases.


Power Spectral Density
0.12
0.1
0.08
0.06
0.04
0.02
0
0
1
2
3
4
5
6
7
8
9
10
f
kp=1
kp=2
kp=5
kp=10


Power Spectral density (for kp=0.1)
x 10^-3
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
8
9
10
f


Power Spectral Density (for kp=0.01)
x 10^-5
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
8
9
10
f