EE160 Handout #18
February 27, 2007

Solutions to HW #6

1. Let I_n take values -1 and 1 corresponding to symbols 0 and 1 at time n. If we define $g(t)$ as follows

$$g(t) = \begin{cases} A & 0 \le T_b/2 \\ -A & T_b/2 \le t \le T_b \\ 0 & \text{otherwise} \end{cases}$$

then the transmitted signal is $s(t) = \sum_n I_n g(t - nT_b)$. We need to first find $R_s(t, \tau)$. Note that $EI_nI_m = \delta(m,n)$ since the symbols are randomly generated and are equally likely.

$$\begin{aligned} R_s(t,\tau) &= E\sum_m\sum_n I_nI_m g(t-nT_b)g(t+\tau-mT_b) \\ &= \sum_m\sum_n E(I_nI_m)g(t-nT_b)g(t+\tau-mT_b) \\ &= \sum_m\sum_n \delta(m,n)g(t-nT_b)g(t+\tau-mT_b) \\ &= \sum_n g(t-nT_b)g(t+\tau-nT_b) \end{aligned}$$

This is a cyclostationary process, i.e. periodic in t with period T_b and hence we have

$$\begin{aligned} R_s(\tau) &= \frac{1}{T_b}\sum_n \int_{-T_b/2}^{T_b/2} g(t-nTb)g(t+\tau-nT_b)dt \\ &= \frac{1}{T_b}\sum_n \int_{-T_b/2-nT_b}^{T_b/2-nT_b} g(t)g(t+\tau)dt \\ &= \frac{1}{T_b}\int_{-\infty}^{\infty} g(t)g(t+\tau)dt \\ &= \frac{1}{T_b}R_g(\tau) \end{aligned}$$

Hence the power spectral density is the fourier transform of the above i.e. $\frac{1}{T_b}S_g(f)$. This is simply $\frac{1}{T_b}|G(f)|^2$ where $g(t)$ is as defined above. Therefore

$$\begin{aligned} S(f) &= \frac{1}{T_b}\left|\frac{AT_b}{2}\,\text{sinc}\left(\frac{fT_b}{2}\right)e^{-j2\pi fT_b/4} - \frac{AT_b}{2}\,\text{sinc}\left(\frac{fT_b}{2}\right)e^{-j2\pi f3T_b/4}\right|^2 \\ &= A^2T_b\,\text{sinc}^2\left(\frac{fT_b}{2}\right)\sin^2\left(\frac{\pi fT_b}{2}\right) \end{aligned}$$

2. (a) We will show this by induction. For $n = 1$ i.e. the first decision, we have a probability p_1 of error. Clearly $p_1 = \frac{1}{2}[1-(1-2p_1)^1]$.
   Let p_k be as specified. For $n = k+1$ we have

$$\begin{aligned} p_{k+1} &= p_k(1-p_1) + (1-p_k)p_1 \\ &= (1-p_1)\frac{1}{2}[1-(1-2p_1)^k] + p_1(1-\frac{1}{2}[1-(1-2p_1)^k]) \quad \text{by induction hypothesis} \\ &= (1-2p_1)^k(-\frac{1}{2}(1-p_1)+p_1) + \frac{1}{2}(1-p_1) + \frac{1}{p_1} \\ &= (1-2p_1)^k(-\frac{1}{2}+p_1) + \frac{1}{2} \\ &= \frac{1}{2}[1-(1-2p_1)^{k+1}] \end{aligned}$$

   This is of the specified form.

   (b) Since p is very small we have $(1-2p_1)^n \approx 1-2np_1$ (since $(1+x)^n \approx 1+nx$ for $|x| \ll 1$). Therefore $p_n \approx \frac{1}{2}[1-(1-2np_1)] = np_1$ which is in the range $[0,1]$ since n is not too large.

3. Let $m_i \;\; i = 1,2,\cdots,L+1$ denote the end points of the intervals. Hence $m_1 = -A$ and $m_{L+1} = A$. Let the quantization levels be set at $v_i \;\; i = 1,2,\cdots,L$ where $v_i \in [m_i, m_{i+1})$. For any distribution, such a setup implies that $m_i = \frac{v_{i-1}+v_i}{2}$. In addition, since we have a uniform distribution 3.45 simply gives $v_k = \frac{m_k+m_{k+1}}{2}$. Together we have the optimal quantizer given by the uniform quantizer i.e. $m_i = -A+(i-1)W$ and $v_i = m_i + \frac{W}{2}$ where $W = \frac{2A}{L}$.

4. (a) The 8 intervals are encoded with the binary representation of $0,\cdots,7$ in order. Consider any interval. Let it correspond to the bit sequence $b_3b_2b_1$ i.e. we quantize any point in this interval to $b_1b_2b_3$. If $b_i \;\; i = 1,2,3$ is in error, we decode to the center of an incorrect interval. This gives a square error distortion of $\int_0^W (x-v)^2 dx$ where $v = (2^i+1)W/2$. (For each interval and each i you can verify that this v gives the intervals we have ended up in after the bit error.) Since we are asked to ignore the possibility of more than 2 bit-errors, these are the only bit errors we have to consider. We may assume that the probability of each of these is $P_e/3$. This gives us a square error distortion of $\frac{85}{12}W^3P_e$. In addition, if we receive the correct bit sequence (with probability $1-P_e$) we incur a square error distortion of $\frac{1}{12}W^3$. Together, we have a distortion of $\frac{85}{12}W^3P_e + (1-P_e)\frac{1}{12}W^3$.

   (b) The encoding 000, 001, 010, 011, 100, 101, 110, 111 of the 8 intervals in order is a (possible) Gray encoding.

   (c) For the encoding given above, we need to do calculations similar to those in (a). However, because of the ordering, we do not have a generic expression for the intervals we might end up in case of a bit error. For instance, if 010 is being sent (corresponding to a point in the 3^{rd} interval), we could end up in the $2^{nd}, 4^{th}$ or the 7^{th} interval. After some calculation, we get a distortion of $\frac{109}{12}W^3P_e + (1-P_e)\frac{1}{12}W^3$ which is more than that in (a).

5. By symmetry of the pdf we will have symmetric intervals on the positive and negative m-axis. Let us consider only the positive axis. If 0, a and 1 are the end points of the

quantization intervals, the optimum quantization levels can be considered to be $a - x$ and $a + x$. 3.45 now gives the following equations

$$\begin{aligned} a + x &= \frac{1 + a - 2a^2}{3(1 - a)} \\ a - x &= \frac{3a - 2a^2}{3(2 - a)} \end{aligned}$$

Solving, we get $a = \frac{3-\sqrt{5}}{2} = 0.3820$ and the quantization levels as 0.1760 and 0.5880. By symmetry we know the intervals and quantization levels on the negative m-axis. The mean square error is 0.01548.
With uniform quantization the mean square error is 0.02083. As expected, the Lloyd-Max quantizer performs better.