Comments on "Performance Analysis of a Deterministic Channel Estimator for Block Transmission Systems With Null Guard Intervals"

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***Abstract*— In the above-mentioned paper a Cramer-Rao bound was derived for the performance of a blind channel estimation algorithm. In this paper an error in the bound is pointed out and corrected. It is observed here that the performance of the said algorithm does not achieve the Cramer-Rao bound.**[1]

In the above paper [1], important work has been done to analyze the algorithm in [2] which solves a blind channel estimation problem. The performance of the algorithm in [2] in high SNR region was shown to be as in (33) of [1]. The Cramer-Rao bound (CRB) of the above mentioned blind estimation problem was shown to be as in (49) of [1]. The coincidence of (33) and (49) led the authors of [1] to claim that the algorithm in [2] is statistically efficient (i.e., achieves the CRB) at high SNR values. However, we have found an error in the derivation of (49), which invalidates this claim. Eq. (49) of [1] was derived from (80) in Appendix B of [1]. The second equality of (80) is not valid in general since it is conditioned on the validity of the matrix identity

$$(\mathbf{A}\mathbf{B}\mathbf{A}^H)^{-1} = \mathbf{A}^{H\dagger}\mathbf{B}^{-1}\mathbf{A}^{\dagger} \quad (1)$$

where $\mathbf{A}$ is a full rank matrix with more columns than rows and $\mathbf{B}$ is a square positive definite matrix. But a simple example shows that this identity is not true in general: set

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

then the left hand side of (1) is $\mathbf{I}_2$ whereas the right hand side is $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

A correction to the CRB, however, is easy to make. The corrected CRB can be simply taken as the first equality of (80) of [1]:

$$\mathbf{C}_{CR} = \sigma_v^2 \left[\tilde{\mathcal{V}} \left[\mathbf{I}_{L\times L} \otimes (\mathbf{F}^*\mathbf{S}_N^*\mathbf{S}_N^T\mathbf{F}^T)\right] \tilde{\mathcal{V}}^H\right]^{-1} \quad (2)$$

(in the original text [1], σ_v^2 appeared in the denominator, which was presumably a typographical error).

We conduct numerical simulations to compare

$$\mathbf{C}_{hh} \approx \sigma_v^2 \tilde{\mathcal{V}}^{\dagger H} \left[\mathbf{I}_{L\times L} \otimes \left(\mathbf{F}^{-T}(\mathbf{S}_N^*\mathbf{S}_N^T)^{-1}\mathbf{F}^{-*}\right)\right] \tilde{\mathcal{V}}^{\dagger}$$

from (33) of [1] and the corrected CRB in (2). The simulation setting basically follows that in [1]: the channel order is chosen as $L = 4$ and the channel coefficients are i.i.d., zero-mean, unit variance complex Gaussian random variables. The data length per block is $M = 12$ and the number of blocks N ranges from 8 to 1000. Elements of the data matrix $\mathbf{S}_N$ were generated using the QPSK constellation and $\mathbf{F}$ is chosen as $\mathbf{I}_M$. One hundred independent realizations of channel coefficients and 10 independent realizations of data blocks $\mathbf{S}_N$ are used (totally 1000 different pairs of $\mathbf{S}_N$ and $\mathbf{h}$). Traces of $\mathbf{C}_{hh}$ and $\mathbf{C}_{CR}$ in (2) are computed for these 1000 realizations and the averages are reported in Table I.

N	$\mathrm{tr}(\mathbf{C}_{hh})/\sigma_v^2$	$\mathrm{tr}(\mathbf{C}_{CR})/\sigma_v^2$	$\frac{\mathrm{tr}(\mathbf{C}_{hh})-\mathrm{tr}(\mathbf{C}_{CR})}{\mathrm{tr}(\mathbf{C}_{CR})}$
8	–	1.7752	–
12	184.01	1.3373	136.6002
14	6.8590	1.0981	5.2462
16	3.5362	0.9760	2.6233
20	1.7197	0.7414	1.3196
100	0.1614	0.1448	0.1147
1000	1.5149×10^{-2}	1.4986×10^{-2}	0.0109

TABLE I
COMPARISON OF EQ. (33) IN [1] AND EQ. (2); THE DATA LENGTH PER BLOCK IS $M = 12$

We find from Table I that there is a significant discrepancy between the corrected CRB in (2) and the performance of the algorithm in [2] (Eq. (33) in [1]), especially when N is small. Furthermore, when $N < M$, the inverse of $\mathbf{S}_N^*\mathbf{S}_N^T$ in (33) of [1] does not exist, but $\mathbf{C}_{CR}$ in (2) still gives a finite value. This suggests there might exist algorithms (e.g., see [4]–[6]) other than [2] which solve the aforementioned blind estimation problem when $N < M$. On the other hand, when N is large, the difference between traces of $\mathbf{C}_{hh}$ and $\mathbf{C}_{CR}$ tends to shrink, but it never goes to zero. This observation is accounted for by the following lemma, where we use notations from the singular value decomposition of the $L \times LM$ full-rank matrix $\tilde{\mathcal{V}}$:

$$\tilde{\mathcal{V}} = \mathbf{U} \begin{bmatrix} \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 \end{bmatrix}^H, \quad (3)$$

where $\mathbf{U}$ is a unitary matrix, $\mathbf{D}$ is a diagonal matrix with positive diagonal entries, and $\mathbf{V} := \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 \end{bmatrix}$ is a unitary matrix. $\mathbf{V}_1$ and $\mathbf{V}_2$ are the first L and the last $(M-1)L$ columns of $\mathbf{V}$, respectively.

Lemma 1: If $N \geq M$, then $\mathrm{tr}(\mathbf{C}_{hh}) \geq \mathrm{tr}(\mathbf{C}_{CR})$, with equality if and only if

$$\mathbf{V}_1^H\mathbf{B}\mathbf{V}_2 = \mathbf{0} \quad (4)$$

where $\mathbf{B} := \mathbf{I}_{L\times L} \otimes (\mathbf{F}^*\mathbf{S}_N^*\mathbf{S}_N^T\mathbf{F}^T)$ and $\mathbf{V}_1$ and $\mathbf{V}_2$ are defined as in (3).

Proof: Since both $\mathbf{C}_{hh}$ and $\mathbf{C}_{CR}$ are positive definite (p.d.), the statement $\mathrm{tr}(\mathbf{C}_{hh}) \geq \mathrm{tr}(\mathbf{C}_{CR})$ is equivalent to the statement that $\mathbf{C}_{hh} - \mathbf{C}_{CR}$ is a positive semi-definite matrix. We first observe that $\mathbf{B}$ is p.d. since $\mathbf{F}^*\mathbf{S}_N^*\mathbf{S}_N^T\mathbf{F}^T$ is p.d. Recall the SVD of $\tilde{\mathcal{V}}$ as in (3) where $\mathbf{U}$ and $\mathbf{V} := [\mathbf{V}_1, \mathbf{V}_2]$ are unitary matrices and $\mathbf{D}$ is a diagonal matrix with positive diagonal entries. Define $\mathbf{B}_2 := \mathbf{V}^H\mathbf{B}\mathbf{V}$ which is obviously also p.d. Partition $\mathbf{B}_2$ and $\mathbf{B}_2^{-1}$ into

$$\mathbf{B}_2 = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{12}^H & \mathbf{B}_{22} \end{bmatrix} \text{ and } \mathbf{B}_2^{-1} = \begin{bmatrix} \mathbf{B}'_{11} & \mathbf{B}'_{12} \\ \mathbf{B}'^H_{12} & \mathbf{B}'_{22} \end{bmatrix},$$

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respectively, so that $\mathbf{B}_{11}$ and $\mathbf{B}'_{11}$ have the same size as $\mathbf{D}$ ($L \times L$). Then we have

$$\begin{aligned} \mathbf{C}_{CR} &= \sigma_v^2(\tilde{\mathcal{V}}\mathbf{B}\tilde{\mathcal{V}}^H)^{-1} = \sigma_v^2\mathbf{U}([\mathbf{D}\ \ \mathbf{0}]\mathbf{B}_2[\mathbf{D}\ \ \mathbf{0}]^T)^{-1}\mathbf{U}^H \\ &= \sigma_v^2\mathbf{U}\mathbf{D}^{-1}\mathbf{B}_{11}^{-1}\mathbf{D}^{-1}\mathbf{U}^H \end{aligned}$$

and

$$\begin{aligned} \mathbf{C}_{hh} &= \sigma_v^2\tilde{\mathcal{V}}^{\dagger H}\mathbf{B}^{-1}\tilde{\mathcal{V}}^{\dagger} = \sigma_v^2\mathbf{U}[\mathbf{D}^{-1}\ \ \mathbf{0}]\mathbf{B}_2^{-1}[\mathbf{D}^{-1}\ \ \mathbf{0}]^T\mathbf{U}^H \\ &= \sigma_v^2\mathbf{U}\mathbf{D}^{-1}\mathbf{B}'_{11}\mathbf{D}^{-1}\mathbf{U}^H. \end{aligned}$$

So $\mathbf{C}_{CR} \leq \mathbf{C}_{hh}$ if and only if

$$\mathbf{B}_{11}^{-1} \leq \mathbf{B}'_{11} = \mathbf{B}_{11}^{-1} + \mathbf{B}_{11}^{-1}\mathbf{B}_{12}\mathbf{\Delta}_{B11}^{-1}\mathbf{B}_{12}^H\mathbf{B}_{11}^{-1}$$

where $\mathbf{\Delta}_{B11} := \mathbf{B}_{22} - \mathbf{B}_{12}^H\mathbf{B}_{11}^{-1}\mathbf{B}_{12}$ is the Schur complement [3] of $\mathbf{B}_{11}$ in $\mathbf{B}_2$. Since $\mathbf{B}_2$ is p.d., both $\mathbf{B}_{11}$ and $\mathbf{\Delta}_{B11}$ are also p.d. (see theorem (7.7.6) of [3]). So $\mathbf{B}_{11}^{-1} \leq \mathbf{B}'_{11}$ is readily verified, with equality if and only if $\mathbf{B}_{12} = \mathbf{0}$, which is equivalent to (4). ■

Using Lemma 1, we find that (33) in [1] achieves the CRB if and only if (4) is satisfied. Eq. (4) can be satisfied only in one of two possible ways described as follows.

a) If $\mathbf{B}$ is the identity matrix or a positive multiple thereof, i.e., $\mathbf{S}_N^*\mathbf{S}_N^T = c\mathbf{I}_M$ for some positive constant c, then Eq. (4) is satisfied. This is extremely unlikely to happen since elements of $\mathbf{S}_N$ are i.i.d. random symbols. However, we should note that $(1/N)\mathbf{S}_N^*\mathbf{S}_N^T$ tends to approach $c\mathbf{I}_M$ for some $c > 0$ as N goes to infinity. This explains to some extent why the discrepancy between tr$(\mathbf{C}_{hh})$ and tr$(\mathbf{C}_{CR})$ approaches zero as $N \to \infty$.
b) On the other hand, if $\mathbf{B} \neq c\mathbf{I}$, then columns of $\mathbf{V}_1$ and $\mathbf{V}_2$ must match the eigenvectors of $\mathbf{B}$ in order to make (4) true. But this is also extremely unlikely since $\tilde{\mathcal{V}}$ depends on, besides $\mathbf{S}_N$, the random channel coefficients which we have no control of.

In conclusion, the gap existing between (33) of [1] and the corrected CRB (2) suggests that there might exist algorithms other than [2] which yield a better performance than [2] in high-SNR region. Indeed there are such algorithms as reported in [4]–[6].

References

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