NECESSARY AND SUFFICIENT CONDITIONS FOR LTI SYSTEMS TO PRESERVE SIGNAL RICHNESS

Borching Su and P. P. Vaidyanathan

Dept. of Electrical Engineering
California Institute of Technology, Pasadena, CA 91125, USA
E-mail: borching@systems.caltech.edu, ppvnath@systems.caltech.edu

ABSTRACT

There are many ways to define richness of a discrete time signal. In this paper we consider a particular definition and explore the conditions under which a linear time invariant (LTI) system preserves the richness property. A set of necessary and sufficient conditions has been found. Using this, paraunitary and unimodular matrices can be shown not to preserve richness unless they are constant matrices (or a delayed version in the paraunitary case). A structured proof of the necessary and sufficient conditions is also presented.[1]

1. INTRODUCTION

In some applications a sequence of $M \times 1$ vectors $\mathbf{x}(n), n \geq 0$ is defined to be **rich** or **rank-rich** if the matrix

$$\begin{bmatrix} \mathbf{x}(0) & \mathbf{x}(1) & \cdots & \mathbf{x}(K_x) \end{bmatrix}$$

has rank M for sufficiently large K_x [1]. This property is important, for example, when we try to identify an unknown communication channel from output measurements alone using filter bank precoders [3]. Now, signals are sometimes preconditioned by linear transformations before they are used in such an application [4]. This leads us to explore the conditions under which the linear precoders will preserve richness of the vectorized signals. In [1] some nontrivial sufficient conditions have been presented. Specific examples given in [1] also show that paraunitary and unimodular matrices do not in general preserve richness. In this paper, we will present the most general necessary and sufficient conditions for preserving richness.

Let the linear time invariant system be characterized by the $M \times M$ polynomial matrix

$$\mathbf{H}(z) = \sum_{k=0}^{N} \mathbf{h}(k) z^{-k}$$

so that

$$\mathbf{y}(n) = \sum_{k=0}^{N} \mathbf{h}(k)\mathbf{x}(n-k).$$

We say the system $\mathbf{H}(z)$ is **richness-preserving** (RP) if for any rank-rich input $\mathbf{x}(n)$, the output $\mathbf{y}(n)$ is also rank-rich. A set of necessary and sufficient conditions will be presented in Sec. 2. In Sec. 3 we will show that paraunitary (PU) matrices and unimodular matrices cannot satisfy the necessary conditions unless they are constant matrices (with a possible delay in the PU case). The proof of the main theorem will be given in Sec. 4. Throughout the paper we will use the term richness to imply rank-richness.

[1] Work supported in parts by the NSF grant CCF-0428326 and the Moore Fellowship of the California Institute of Technology.

1.1. Notations

Boldfaced lower case letters represent column vectors, and boldfaced upper case letters are reserved for matrices. Superscripts as in $\mathbf{A}^T$ and $\mathbf{A}^H$ denote the transpose and transpose-conjugate operations, respectively, of a matrix or a vector. $\widetilde{\mathbf{H}}(z)$ represents $\mathbf{H}^H(1/z^*)$, and $[\mathbf{v}]_i$ denotes the ith element of vector $\mathbf{v}$.

2. MAIN THEOREM AND EXAMPLES

In this section we will describe the necessary and sufficient conditions for an LTI system to preserve richness. The proof of the theorem will be given in Sec. 4.

Theorem 1: An Nth order, $M \times M$ polynomial matrix

$$\mathbf{H}(z) = \sum_{k=0}^{N} \mathbf{h}(k) z^{-k}$$

is a **richness-preserving (RP)** LTI system **if and only if** either one of the following conditions is true:

(a) There exist a nonsingular $M \times M$ matrix $\mathbf{A}$ and constants $g_0, g_1, \cdots, g_N$ of which at least one is nonzero such that $\mathbf{h}(k) = g_k \mathbf{A}$.

(b) There exist a nonzero row vector $\mathbf{v}^T$ and a set of column vectors $\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_N$ such that $\mathbf{h}(k) = \mathbf{a}_k \mathbf{v}^T$ for any k, and $\begin{bmatrix} \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \mathbf{a}_N \end{bmatrix}$ has full rank M. $\square$

It is obvious that conditions (a) and (b) cannot be satisfied at the same time. We can hence say there are two types of RP matrices, namely, Type A and Type B, according to the statement of Theorem 1. For Type A matrices, each nonzero coefficient matrix is nonsingular while for Type B matrices, each nonzero coefficient matrix has unit rank. Notice in particular that the order N and the size M of a Type B matrix must satisfy $N \geq M-1$ to meet the full rank criterion of $\begin{bmatrix} \mathbf{a}_0 & \mathbf{a}_1, & \cdots & \mathbf{a}_N \end{bmatrix}$.

Some special cases have already been noticed in [1]. An Nth order FIR system with the form

$$\mathbf{H}(z) = \mathbf{A}\left(g_0 + g_1 z^{-1} + \cdots + g_N z^{-N}\right)$$

is sufficient to preserve richness if $\mathbf{A}$ is a nonsingular matrix and $g_0 \neq 0$. This is an example of Type A matrices. It had also been shown that, for a first order RP matrix $\mathbf{H}(z) = \mathbf{h}(0) + \mathbf{h}(1)z^{-1}$, $\mathbf{h}(1)$ is necessarily $g_1\mathbf{h}(0)$ for some constant g_1 if $\mathbf{h}(0)$ is nonsingular. A useful corollary of Theorem 1 is as follows:

Corollary 1: Consider the Nth order, $M \times M$ FIR system $\mathbf{H}(z) = \sum_{k=0}^{N} \mathbf{h}(k) z^{-k}$ and assume $\mathbf{h}(0)$ is nonsingular. Then $\mathbf{H}(z)$ is RP **if and only if** there exist a nonsingular $M \times M$ matrix $\mathbf{A}$ and constants $g_0, g_1, \cdots, g_N$ where $g_0 \neq 0$ such that $\mathbf{h}(k) = g_k \mathbf{A}$. □

For RP matrices where $\mathbf{h}(0)$ is singular, a simple example has also been given in [1]:

$$\mathbf{H}(z) = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} + z^{-1} \begin{bmatrix} 0 & 0 \\ 1 & a \end{bmatrix}.$$

This is an example of Type B matrices. In this example, each row of all coefficient matrices is proportional to row vector $\mathbf{v}^T = \begin{bmatrix} 1 & a \end{bmatrix}$. Column vectors $\mathbf{a}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $\mathbf{a}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ are used in this example. According to condition (b) of Theorem 1, one can generate a Type B RP matrix by arbitrarily picking up a nonzero row vector $\mathbf{v}^T$ and column vectors $\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_N$ where $\begin{bmatrix} \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \mathbf{a}_N \end{bmatrix}$ has full rank. For an RP matrix, if $\mathbf{h}(0)$ is singular but nonzero, it must be a Type B matrix. Another corollary of Theorem 1 is as follows:

Corollary 2: Consider the Nth order, $M \times M$ FIR system $\mathbf{H}(z) = \sum_{k=0}^{N} \mathbf{h}(k) z^{-k}$ with size $M \times M$ and assume $\mathbf{h}(0) \neq \mathbf{0}$ is singular. Then $\mathbf{H}(z)$ is RP **if and only if** there exist a nonzero row vector $\mathbf{v}^T$ and $N+1$ column vectors $\mathbf{a}_0, \mathbf{a}_1, \cdots \mathbf{a}_N$ such that $\mathbf{h}(n) = \mathbf{a}_n \mathbf{v}^T$, $\begin{bmatrix} \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \mathbf{a}_N \end{bmatrix}$ has full rank, and $\mathbf{a}_0 \neq \mathbf{0}$. □

The proofs of the preceding two corollaries will be automatically covered when we prove Theorem 1 in Sec. 4. In these corollaries we have not considered the case where $\mathbf{h}(0) = \mathbf{0}$. If this is true, however, $\mathbf{H}(z)$ is simply a delayed version of another LTI system whose first coefficient is nonzero. Since $\mathbf{H}(z)$ is RP if and only if $z^{-m}\mathbf{H}(z)$ is RP for any m, the assumption $\mathbf{h}(0) \neq \mathbf{0}$ is not a loss of generality.

3. PARAUNITARY AND UNIMODULAR MATRICES

We now present some applications for Theorem 1. We say $\mathbf{H}(z)$ is a paraunitary (PU) matrix if $\widetilde{\mathbf{H}}(z)\mathbf{H}(z) = \mathbf{I}_M$ [2]. A causal system $\mathbf{H}(z)$ is said to be unimodular if $\det(\mathbf{H}(z)) = 1$, so that the inverse of $\mathbf{H}(z)$ is still a causal FIR system [2]. Using Theorem 1, we can show that paraunitary and unimodular matrices cannot preserve richness unless they are constant matrices (with a possible delay in the PU case).

Corollary 3: If a paraunitary matrix $\mathbf{H}(z)$ is richness-preserving, then $\mathbf{H}(z)$ is a constant unitary matrix or a delayed version of it.

Proof: Without loss of generality, assume $\mathbf{h}(0) \neq \mathbf{0}$. Suppose $\mathbf{H}(z) = \sum_{k=0}^{N} \mathbf{h}(k) z^{-k}$ is paraunitary and richness-preserving but not a constant matrix (i.e., $N > 0$ and $\mathbf{h}(N)$ is nonzero). From properties of paraunitary matrices we know both $\mathbf{h}(0)$ and $\mathbf{h}(N)$ are singular [2]. Using Corollary 2 of Theorem 1, there exist row vector $\mathbf{v}^T$ and $N+1$ column vectors $\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_N$ such that $\mathbf{H}(z) = \sum_{k=0}^{N} \mathbf{a}_k \mathbf{v}^H z^{-k}$. So $\widetilde{\mathbf{H}}(z) = \sum_{l=0}^{N} \mathbf{v}\mathbf{a}_l^H z^l$ and

$$\widetilde{\mathbf{H}}(z)\mathbf{H}(z) = \sum_{k=0}^{N} \sum_{l=0}^{N} \mathbf{v}\mathbf{a}_l^H \mathbf{a}_k \mathbf{v}^H z^{-(k-l)}.$$

The constant term (z^0) of $\widetilde{\mathbf{H}}(z)\mathbf{H}(z)$ would be

$$\sum_{k=0}^{N} \mathbf{v}\mathbf{a}_k^H \mathbf{a}_k \mathbf{v}^H = \left[\sum_{k=0}^{N} \mathbf{a}_k^H \mathbf{a}_k \right] \mathbf{v}\mathbf{v}^H$$

since $\mathbf{a}_k^H \mathbf{a}_k$ are all constants. The matrix $\mathbf{v}\mathbf{v}^H$ obviously has rank one. This contradicts $\widetilde{\mathbf{H}}(z)\mathbf{H}(z) = \mathbf{I}_M$, completing the proof. □

Corollary 4: If a unimodular matrix $\mathbf{H}(z)$ is richness-preserving, then $\mathbf{H}(z)$ is a constant matrix.

Proof: If $\mathbf{H}(z) = \sum_{k=0}^{N} \mathbf{h}(k) z^{-k}$ is unimodular, $\det(\mathbf{h}(0)) = \det(\mathbf{H}(\infty)) = 1$. So $\mathbf{h}(0)$ must be nonsingular. If $\mathbf{H}(z)$ is also RP, it must satisfy condition (a) in Theorem 1. Then $\mathbf{H}(z) = \left(\sum_{k=0}^{N} g_k z^{-k} \right) \mathbf{A}$ and $\det(\mathbf{H}(z)) = \left(\sum_{k=0}^{N} g_k z^{-k} \right)^M \det(\mathbf{A}) = 1$, where $\mathbf{A}$ is nonsingular. So we have $g_k = 0$ for $k > 0$ and hence $\mathbf{H}(z)$ must be a constant matrix. □

4. PROOF OF THE MAIN THEOREM

4.1. Sketch of the Proof

In this section, we will prove Theorem 1 step by step. We will first show that conditions (a) and (b) are sufficient. Then we will describe several lemmas for proof of necessity. A term *coefficient rank* will be defined to denote the ranks of all nonzero coefficient matrices since they will prove to be the same. The coefficient rank will later on prove to be either unity or M. Finally, for the case of unity coefficient rank, we will show condition (b) is necessary, and for the case of full coefficient rank, condition (a) is necessary.

4.2. Proof of Sufficiency

We first prove conditions (a) and (b) in Theorem 1 are sufficient for preserving richness.

Proof: If $\mathbf{H}(z)$ satisfies condition (a), by Theorem 1 in [1], it is RP. Suppose $\mathbf{H}(z)$ satisfies condition (b) but is not RP. Then there exists a rich input $\mathbf{x}(n)$ such that the output $\mathbf{y}(n)$ is not rich, i.e., there exists a row vector $\mathbf{w}^T$ such that $\mathbf{w}^T \mathbf{y}(n) = 0, \forall n$. Using $\mathbf{y}(n) = \sum_{k=0}^{N} \mathbf{h}(k) \mathbf{x}(n-k)$, we have the following equations:

$$\begin{aligned} (\mathbf{w}^T \mathbf{a}_0)(\mathbf{v}^T \mathbf{x}(0)) &= 0 \\ (\mathbf{w}^T \mathbf{a}_0)(\mathbf{v}^T \mathbf{x}(1)) + (\mathbf{w}^T \mathbf{a}_1)(\mathbf{v}^T \mathbf{x}(0)) &= 0 \\ &\vdots \\ \sum_{k=0}^{N} (\mathbf{w}^T \mathbf{a}_k)(\mathbf{v}^T \mathbf{x}(N-k)) &= 0. \end{aligned}$$

If $\mathbf{v}^T \mathbf{x}(0)$ is not zero, then from the first equation we have $\mathbf{w}^T \mathbf{a}_0 = 0$. Substituting this into the second equation, we get

$$(\mathbf{w}^T \mathbf{a}_1)(\mathbf{v}^T \mathbf{x}(0)) = 0.$$

So $\mathbf{w}^T \mathbf{a}_1$ has to be zero. Repeat these substitutions and we will have $\mathbf{w}^T \mathbf{a}_k = 0, \forall k, 0 \leq k \leq N$. This contradicts the statement that $[\mathbf{a}_0, \mathbf{a}_1, ..., \mathbf{a}_N]$ has rank M. So $\mathbf{v}^T \mathbf{x}(0)$ has to be zero. Substituting this into all equations and repeating the same derivations, we will have $\mathbf{v}^T \mathbf{x}(1) = \mathbf{0}$ as well. Repeating this we get $\mathbf{v}^T \mathbf{x}(n) = 0$ for all n. This violates richness of the input $\mathbf{x}(n)$. So condition (b) is also sufficient. □

4.3. Lemmas for Proof of Necessity

Lemma 1: If an $M \times M$ polynomial matrix $\mathbf{H}(z) = \sum_{k=0}^{N} \mathbf{h}(k) z^{-k}$ is richness-preserving, then there exist an $M \times M$ constant matrix $\mathbf{A}$ and $M \times M$ diagonal matrices $\mathbf{D}_k$ such that $\mathbf{h}(k) = \mathbf{D}_k \mathbf{A}$.

Proof: For $0 \leq k \leq N$, we assume

$$\mathbf{h}(k) = \begin{bmatrix} \mathbf{a}_{1k} & \mathbf{a}_{2k} & \cdots & \mathbf{a}_{Mk} \end{bmatrix}^T,$$

where

$$\mathbf{a}_{ik}^T = \begin{bmatrix} a_{i1k} & a_{i2k} & \cdots & a_{iNk} \end{bmatrix}$$

is the ith row of $\mathbf{h}(k)$. Focusing on the ith row of $\mathbf{H}(z)$, we use

$$\mathbf{b}_k^T = \begin{bmatrix} b_{1k} & b_{2k} & \cdots & b_{Nk} \end{bmatrix}$$

to denote $\mathbf{a}_{ik}^T$ for simplicity. Since $\mathbf{H}(z)$ is richness preserving, any row of $\mathbf{H}(z)$ cannot be all zeros. So there exists b_{jk} that is nonzero. Without loss of generality, assume $b_{10} \neq 0$. Construct the input as:

$$\begin{aligned} \mathbf{x}(0) &= b_{20}\mathbf{e}_1 - b_{10}\mathbf{e}_2 \\ \mathbf{x}(1) &= b_{21}\mathbf{e}_1 - b_{11}\mathbf{e}_2 \\ &\vdots \\ \mathbf{x}(N) &= b_{2N}\mathbf{e}_1 - b_{1N}\mathbf{e}_2 \\ \mathbf{x}(m(N+1)+k) &= b_{(m+2),k}\mathbf{e}_1 - b_{1k}\mathbf{e}_{m+2}, \\ & \quad 0 \leq m \leq M-2, 0 \leq k \leq N. \end{aligned}$$

For simplicity, we will use $\mathbf{x}_m(k)$ to denote $\mathbf{x}(m(N+1)+k)$ By the definitions above, one can verify the following things for $0 \leq m \leq M-2,\ 0 \leq k, l \leq N$.
(1) $\mathbf{b}_k^T \mathbf{x}_m(k) = 0$.
(2) $\mathbf{b}_k^T \mathbf{x}_m(l) + \mathbf{b}_l^T \mathbf{x}_m(k) = 0$.
Using these results, it can be shown that

$$\begin{aligned} [\mathbf{y}(n)]_i &= \left[\sum_{k=0}^{N} \mathbf{h}(k)\mathbf{x}(n-k) \right]_i \\ &= \sum_{k=0}^{N} \mathbf{b}_k^T \mathbf{x}(n-k) = 0. \end{aligned}$$

Hence the output $\mathbf{y}(n)$ is not rich. Since $\mathbf{H}(z)$ is richness preserving, $\mathbf{x}(n)$ must also be not rich. Define the $M \times M$ matrix

$$\mathbf{X}_1 = \begin{bmatrix} \mathbf{x}(0) & \mathbf{x}(1) & \mathbf{x}_1(0) & \mathbf{x}_2(0) & \cdots & \mathbf{x}_{M-2}(0) \end{bmatrix}.$$

One can verify the absolute value of the determinant of $\mathbf{X}_1$ is $|\det(\mathbf{X}_1)| = |b_{10}|^{M-2} |b_{10}b_{21} - b_{11}b_{20}|$. Since $\mathbf{x}(n)$ is not rich, $\det(\mathbf{X}_1) = 0$. Since b_{10} is nonzero, we get $b_{10}b_{21} = b_{11}b_{20}$, or $b_{21} = d_{i1}b_{20}$, where d_{i1} is chosen as b_{11}/b_{10}. Now we define another $M \times M$ matrix by replacing $\mathbf{x}(1)$ in the definition of $\mathbf{X}_1$ with another $\mathbf{x}_m(1)$, and we obtain $b_{(m+2),1} = d_{i1}b_{(m+2),0}$. These results for all m imply that $\mathbf{b}_1 = d_{i1}\mathbf{b}_0$, or $\mathbf{a}_{i1} = d_{i1}\mathbf{a}_{i0}$.

If we replace $\mathbf{x}(1)$ in the definition of $\mathbf{X}_1$ with $\mathbf{x}_m(k)$, we can show that $\exists d_{ik}$ such that $\mathbf{b}_k = d_{ik}\mathbf{b}_0$, or $\mathbf{a}_{ik} = d_{ik}\mathbf{a}_{i0}$. Finally, define $\mathbf{v}_i = \mathbf{a}_{i0}$ and $d_{i0} = 1$, then we have $\mathbf{a}_{ik} = d_{ik}\mathbf{v}_i$ for all i and k. The reader has to note that here we assign $\mathbf{v}_i$ as $\mathbf{a}_{i0}$ just because of the assumption that b_{10} is nonzero without loss of generality. If $b_{10} = 0$, we can find another b_{jk} that is nonzero and do similar derivation, and $\mathbf{v}_i$ here will be assigned as another $\mathbf{a}_{ik}$ rather than $\mathbf{a}_{i0}$. After all, $\exists d_{ik}, \mathbf{v}_i$ such that $\mathbf{a}_{ik} = d_{ik}\mathbf{v}_i$ is still true for all i and k. Now we simply assign

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_M \end{bmatrix}^T$$

and

$$\mathbf{D}_k = \text{diag} \begin{bmatrix} d_{1k} & d_{2k} & \cdots & d_{Mk} \end{bmatrix}.$$

Then the proof is complete. □

Lemma 1 will play an important role in the proof of necessity for both conditions (a) and (b). Some other useful lemmas will be presented here.

Lemma 2: $\mathbf{H}(z)$ is RP if and only if $\mathbf{A}\mathbf{H}(z)$ is RP, where $\mathbf{A}$ is any nonsingular $M \times M$ matrix.
Proof: This lemma becomes obvious when we recognize that $\mathbf{x}(n)$ is rich if and only if $\mathbf{A}\mathbf{x}(n)$ is rich for any nonsingular matrix $\mathbf{A}$.□

Lemma 3: $\mathbf{H}(z)$ is RP if and only if $z^{-k}\mathbf{H}(z)$ is RP, where k is any nonnegative integer.
Proof: This is self-evident. □

Lemma 2 allows us to do invertible row operations on $\mathbf{H}(z)$ since each invertible row operation corresponds to a nonsingular matrix. Lemma 3 allows us to assume $\mathbf{h}(0) \neq \mathbf{0}$ for an RP matrix $\mathbf{H}(z)$.

4.4. Coefficient Rank of an RP System

Lemma 4: For an FIR system $\mathbf{H}(z) = \sum_{k=0}^{N} \mathbf{h}(k) z^{-k}$ which preserves richness, the ranks of all nonzero coefficient matrices must be the same. We call this value the *coefficient rank* of an RP system.
Proof: Suppose $\mathbf{h}(j)$ has the smallest rank ρ among all nonzero $\mathbf{h}(k)$ ($\rho > 0$). By Lemma 2, we can do invertible row operations on $\mathbf{H}(z)$ such that $\mathbf{h}(j)$ can be expressed as

$$\mathbf{h}(j) = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots \mathbf{v}_\rho & \mathbf{v}_\rho & \cdots & \mathbf{v}_\rho \end{bmatrix}^T$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_\rho$ are linearly independent nonzero column vectors. By Lemma 1, there exist a constant matrix $\mathbf{A}$ and a diagonal matrix $\mathbf{D}_j$ such that $\mathbf{h}(j) = \mathbf{D}_j\mathbf{A}$. Since each row of $\mathbf{h}(j)$ is nonzero, all diagonal entries of $\mathbf{D}_j$ must be nonzero and $\mathbf{A}$ also has rank ρ.

Now for any other nonzero coefficient matrix $\mathbf{h}(k)$, there exists a diagonal matrix $\mathbf{D}_k$ such that $\mathbf{h}(k) = \mathbf{D}_k\mathbf{A}$. So rank($\mathbf{h}(k)$)$\leq$ rank($\mathbf{A}$) $= \rho$. Since $\mathbf{h}(j)$ has the smallest nonzero rank ρ, we have rank($\mathbf{h}(k)$) $= \rho$. □

Lemma 5: The coefficient rank of an RP system can only be unity or M.
Proof: Suppose there exists an RP matrix $\mathbf{H}(z)$ that has a coefficient rank ρ where $2 \leq \rho \leq M-1$. By Lemmas 2 and 3, we can assume $\mathbf{h}(0) \neq \mathbf{0}$ and do invertible row operations on $\mathbf{H}(z)$ such that

$$\mathbf{h}(0) = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_\rho & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}^T.$$

Since $\rho < M$, the last row of $\mathbf{h}(0)$ must be a zero vector. The last rows of other $\mathbf{h}(k)$, however, cannot be all zeros. By Lemma 1, there exist a constant matrix $\mathbf{A}$ and a diagonal matrix $\mathbf{D}_0$ such that $\mathbf{h}(0) = \mathbf{D}_0\mathbf{A}$. By Lemma 4 we know rank($\mathbf{A}$) $= \rho$. So the last row of $\mathbf{A}$, namely $\mathbf{v}_2^T$, must be a linear combination of $\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_\rho^T$.

Since $\rho \geq 2$, we can find an i, $1 \leq i \leq \rho$ such that $\mathbf{a}_i$ and $\mathbf{v}_2$ are linearly independent. For convenience we define $\mathbf{v}_1 = \mathbf{a}_i$. Now we can find a set of linear independent vectors $\mathbf{w}_1, \mathbf{w}_2, \cdots,$ $\mathbf{w}_M$ such that $\mathbf{w}_1$ is orthogonal to $\mathbf{v}_2$, $\mathbf{w}_2$ is orthogonal to $\mathbf{v}_1$, and $\mathbf{w}_3$, $\mathbf{w}_4$, ..., $\mathbf{w}_M$ are orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. (For example, we can let $\mathbf{w}_1 = \mathbf{v}_1 - \frac{\mathbf{v}_1^T\mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2$)

Now we focus on the ith and the Mth rows of $\mathbf{H}(z)$. They are $\mathbf{v}_1^T + \sum_{n=1}^N p_n \mathbf{v}_1^T z^{-n}$ and $\sum_{n=1}^N q_n \mathbf{v}_2^T z^{-n}$, respectively, where $\{q_n\}_{n=1}^N$ are not all zeros. Construct the input sequence as:

$$\begin{aligned}
\mathbf{x}(n) &= \mathbf{w}_{n+3}, 0 \leq n \leq M-3 \\
\mathbf{x}(M-2) &= \mathbf{w}_2 \\
\mathbf{x}(M-2+k) &= q_k\mathbf{w}_1 + p_k\mathbf{w}_2, 1 \leq k \leq N \\
\mathbf{x}(n) &= \mathbf{0}, \forall n \geq M+N-1.
\end{aligned}$$

Then one can verify that $[\mathbf{y}(n)]_i = [\mathbf{y}(n)]_M$ for all n and hence $\mathbf{y}(n)$ is not rich. But the input $\mathbf{x}(n)$ is rich. This contradicts the assumption that $\mathbf{H}(z)$ is RP. So the coefficient rank of $\mathbf{H}(z)$ can only be unity or M. □

4.5. Completion of Proof of Necessity

Now we are ready to prove conditions (a) and (b) are necessary for richness preserving property.

Proof: Let $\mathbf{H}(z)$ be RP. By Lemma 3 we assume $\mathbf{h}(0) \neq \mathbf{0}$. If $\mathbf{h}(0)$ is singular, the coefficient rank of $\mathbf{H}(z)$ must be unity by Lemma 5. So there exist a nonzero row vector $\mathbf{v}^T$ and column vectors $\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_N$ such that $\mathbf{h}(k) = \mathbf{a}_k\mathbf{v}^T$. Now we only need to prove $[\mathbf{a}_0, \mathbf{a}_1, ..., \mathbf{a}_N]$ has full rank. If this is not true, we can find an annihilator $\mathbf{w}^T$ for all $\mathbf{a}_k$. Then no matter what the input is, the output $\mathbf{y}(n)$ will have an annihilator $\mathbf{w}^T$, and thus $\mathbf{H}(z)$ becomes richness-destroying. So $[\mathbf{a}_0, \mathbf{a}_1, ..., \mathbf{a}_N]$ must have rank M and thus condition (b) must be true.

If $\mathbf{h}(0)$ is nonsingular, the coefficient rank of $\mathbf{H}(z)$ must be M. By Lemma 2 we assume $\mathbf{h}(0) = \mathbf{I}$ without loss of generality. Using Lemma 1, $\mathbf{H}(z)$ must have the form

$$\mathbf{H}(z) = \mathbf{I} + \sum_{k=1}^N z^{-k} \text{diag}\left(\begin{bmatrix} a_{k1} & a_{k2} & \cdots & a_{kM} \end{bmatrix}\right).$$

Suppose there exist i, j, k such that $a_{ki} \neq a_{kj}$ and $i \neq j$. Let

$$\begin{aligned}
\mathbf{x}(0) &= \mathbf{e}_i + \mathbf{e}_j \\
\mathbf{x}(n) &= a_{nj}\mathbf{e}_i + a_{ni}\mathbf{e}_j, 1 \leq n \leq N \\
\{\mathbf{x}(n)\}_{n=N+1}^{M+N-2} &= \{\mathbf{e}_k | 1 \leq k \leq M, k \neq i, k \neq j\} \\
\mathbf{x}(n) &= \mathbf{0}, \forall n \geq M+N-1.
\end{aligned}$$

Since $a_{ki} \neq a_{kj}$, One can verify that $\mathbf{x}(n)$ is rich. It is also easy to verify the following things for $1 \leq k \leq N$:
(1) $(\mathbf{e}_i^T - \mathbf{e}_j^T)\mathbf{x}(0) = 0$.
(2) $(a_{ki}\mathbf{e}_i^T - a_{kj}\mathbf{e}_j^T)\mathbf{x}(k) = 0$.
(3) $(\mathbf{e}_i^T - \mathbf{e}_j^T)\mathbf{x}(k) + (a_{ki}\mathbf{e}_i^T - a_{kj}\mathbf{e}_j^T)\mathbf{x}(0) = 0$.
(4) $(a_{li}\mathbf{e}_i^T - a_{lj}\mathbf{e}_j^T)\mathbf{x}(k) + (a_{ki}\mathbf{e}_i^T - a_{kj}\mathbf{e}_j^T)\mathbf{x}(l) = 0, 1 \leq l \leq N$.
(5) $(a_{ki}\mathbf{e}_i^T - a_{kj}\mathbf{e}_j^T)\mathbf{x}(l) = 0, N+1 \leq l \leq M+N-2$.

Using these facts, we can show $[\mathbf{y}(n)]_i = [\mathbf{y}(n)]_j, n \geq 0$ and hence $\mathbf{y}(n)$ is not rich. So in order to let $\mathbf{H}(z)$ preserve richness, $a_{ki} = a_{kj}$ must be true for any $i \neq j$ and any k. This means each coefficient matrix of $\mathbf{H}(z)$ is proportional to identity matrix and hence condition (a) must be true. □

The proof of Theorem 1 is now complete.

5. CONCLUDING REMARKS AND OPEN ISSUES

It would be interesting to consider variations in the definition of richness. For example, a variation would be this: $\mathbf{x}(n)$ is rich if for any initial time n_0 there exists an integer K_{n_0} such that

$$\begin{bmatrix} \mathbf{x}(n_0) & \mathbf{x}(n_0+1) & \cdots & \mathbf{x}(n_0+K_{n_0}) \end{bmatrix}$$

has full rank. This appears to be a more practical definition for richness. However, conditions for preserving richness according to this definition could be different from what we have found in this paper. For example, $\mathbf{H}(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ obviously preserves richness according to the old definition. It does not, however, preserve richness according to the new definition. To see this, let $\mathbf{x}(2n) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $\mathbf{x}(2n+1) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ for all nonnegative n. Then the output would be $\mathbf{y}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $\mathbf{y}(n) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ for any positive n. Here the input $\mathbf{x}(n)$ is rich according to both old and new definitions. But the output $\mathbf{y}(n)$ is not rich according to the new definition. The exact sufficient and necessary conditions for preserving richness according to the new definition are not known at the time of writing this paper.

Another issue of interest is the evaluation of the probability for an LTI system to preserve richness. For an LTI system that does not satisfy necessary conditions in Theorem 1, we can manage to find a rich input sequence such that the output of the system is not rich. In practical applications, however, the probability of appearance of such input could almost be zero! This suggests there may exist some LTI systems that, although not satisfying necessary conditions of Theorem 1, still preserve richness with probability one. These systems would still be very useful in practical applications. Finding conditions for such systems under different input statistics could be a challenging but important problem.

6. REFERENCES

[1] Vaidyanathan, P. P. and Su, B., "Staying rich: LTI systems which preserve signal richness" *submitted to* IEEE International Conference on Acoustics, Speech, and Signal Processing March 2005.

[2] Vaidyanathan, P. P. *Multirate systems and filter banks,* Prentice-Hall, 1993.

[3] Scaglione, A., Giannakis, G. B., and Barbarossa, S. "Redundant filter bank precoders and equalizers Part II: Synchronization and direct equalization", IEEE Trans. Signal Processing, vol. 47, pp. 2007-2002, July 1999.

[4] Phoong, S.-M., Chang, K.-Y., and Lin, Y.-P., "Antipodal paraunitary precoding for OFDM application", Proc. IEEE International Symposium on Circuits And Systems, Vancouver, Canada, May 2004.