Generalized Signal Richness Preservation Problem and Vandermonde-Form Preserving Matrices

Borching Su, *Student Member, IEEE*, and P. P. Vaidyanathan, *Fellow, IEEE*

***Abstract*—In this paper, a theoretical problem arising in digital communications, namely the generalized signal richness preservation problem, is addressed and studied. In order to solve the problem, a special class of square matrices, namely the "Vandermonde-form preserving" (VFP) matrices, is introduced and found to be highly relevant to the problem. Several properties of VFP matrices are studied in detail. The necessary and sufficient conditions of the problem have been found, and a systematic proof is also presented.**

***Index Terms*—Blind identification, greatest common divisor, matrix theory, signal richness.**

I. Introduction

IN DIGITAL communications, blind channel identification has been studied in the literature for a considerable period [11]–[14]. Almost every blind identification method assumes a special kind of redundancy in the input signal that facilitates blind identification. In particular, a method using linear redundant precoders with zero padding (ZP), proposed by Scaglione *et al.*[1], assumes the input signal to be *rich*. That is, for a sequence of $M \times 1$ vectors $\mathbf{s}(n), n \geq 0$, there exists a finite integer J such that the $M \times J$ matrix

$$[\mathbf{s}(0) \quad \mathbf{s}(1) \quad \cdots \quad \mathbf{s}(J-1)]$$

has full rank. Now, in some applications, the input signals are usually preconditioned by a linear transformation before being sent to the channel [9]. We are thus interested in whether the signal richness property is preserved after the linear transform. A theoretical treatment of the richness preservation problem has been presented in [10].

More recently, Manton *et al.*proposed another blind identification algorithm for transmitters using ZP that imposes less stringent conditions on input signals [2], [3], requiring only the *coprimality* property. We propose in a companion paper [5]a generalized algorithm of which both blind identification methods mentioned above are special cases. The algorithm requires a generalized definition on signal richness with a parameter Q. When $Q = 1$, it reduces to the conventional definition of richness. When $Q = M - 1$, it becomes equivalent to the coprimality property stated in [3]. Under this new definition of signal richness, the conditions on the linear precoders which preserve signal richness become a different problem. In order to find a guideline to choose precoders in these new applications, we are motivated to find out these conditions.

In this paper, we will focus on the theoretical issues of the generalized signal richness preservation problem and find out the necessary and sufficient conditions for linear precoders to preserve generalized signal richness. The rest of the paper is organized as follows. In Section II we give a definition of generalized signal richness and briefly describe several important properties thereof. Some examples will be given to clarify these properties. In Section III, we will address the problem of preserving generalized signal richness. In Section IV, the class of Vandermonde-form preserving (VFP) matrices will be introduced, and several properties of VFP matrices will be studied in detail. In Section V, the necessary and sufficient conditions for linear precoders to preserve generalized richness will be presented. In Section VI some deeper issues on $(1/Q)$-richness will be studied. For example, the relation between such richness and the "rank" of a signal is studied. Finally, Section VII gives the conclusion and possible future directions. Some results of this paper have been presented in a conference [15].

A. Notations

Boldfaced lower case letters represent column vectors. Boldfaced upper case letters are reserved for matrices. Superscripts as in $\mathbf{A}^T$ and $\mathbf{A}^\dagger$ denote the transpose and transpose-conjugate operations, respectively, of a matrix or a vector. $[\mathbf{v}]_i$ denotes the ith element of vector $\mathbf{v}$, $[\mathbf{A}]_i$ denotes the ith row of matrix $\mathbf{A}$, and $[\mathbf{A}]_{ij}$ denotes the entry at the ith row and the jth column of matrix $\mathbf{A}$. Column and row indices of all vectors and matrices begin at one. $\mathbf{e}_{i,M}$ denotes the ith column of the identity matrix $\mathbf{I}_M$ and is often abbreviated as $\mathbf{e}_i$ when there is no ambiguity about the value of M. All the vectors and matrices in this paper are complex-valued. If $\mathcal{A}$ and $\mathcal{B}$ are multisets (a multiset is like a set, but it may contain identical elements repeated a finite number of times [6]), $\mathcal{A} \biguplus \mathcal{B}$, $\mathcal{A} \bigcap \mathcal{B}$, and $\mathcal{A} \bigcup \mathcal{B}$ denote the multisets defined as follows: if an element occurring exactly a times in $\mathcal{A}$ and b times in $\mathcal{B}$, it occurs exactly $a + b$ times in $\mathcal{A} \biguplus \mathcal{B}$, exactly $\min(a, b)$ times in $\mathcal{A} \bigcap \mathcal{B}$, and exactly $\max(a, b)$ times in $\mathcal{A} \bigcup \mathcal{B}$. A matrix $\mathbf{T}$ is said to be a *Toeplitz matrix* if $\mathbf{T}$ has constant values along diagonals, i.e., $[\mathbf{T}]_{ij} = [\mathbf{T}]_{i+k,j+k}$ for all i, j, k such that the indices of $\mathbf{T}$ in the above equation are within the size of $\mathbf{T}$. A matrix $\mathbf{H}$ is said to be a *Hankel matrix* if $\mathbf{H}$ has constant values along all skew diagonals, i.e.,

Manuscript received April 4, 2006; revised July 3, 2006. This work was supported in part by the National Science Foundation under Grant CCF-0428326, in part by the Office of Naval Research under Grant N00014-06-1-0011, and in part by the Moore Fellowship of the California Institute of Technology. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Kostas Berberidis.

The authors are with the California Institute of Technology, Pasadena, CA 91125 USA (e-mail: borching@caltech.edu; ppvnath@systems.caltech.edu).

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Digital Object Identifier 10.1109/TSP.2007.892712

$[\mathbf{H}]_{ij} = [\mathbf{H}]_{i+k,j-k}$ for all i, j, k such that the indices of $\mathbf{H}$ in the above equation are within the size of $\mathbf{H}$. For example,

$$\mathbf{T} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

is a Toeplitz matrix and

$$\mathbf{H} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 3 \\ 0 & 1 & 3 & 0 \end{bmatrix}$$

is a Hankel matrix.

II. Generalized Signal Richness

A. Definition of Generalized Signal Richness

Definition 1: A sequence of $M \times 1$ vectors $\mathbf{s}(n)$, $n \geq 0$, over the field $\mathbb{C}$ is said to be *rich* if there exists a finite integer J such that the $M \times J$ matrix

$$[\mathbf{s}(0) \quad \mathbf{s}(1) \quad \cdots \quad \mathbf{s}(J-1)]$$

has full row rank M. ■

The definition of the *generalized signal richness* for an $M \times 1$ signal will be given in Definitions 2 and 3 as follows. We first build up the definition of a notation $\mathbf{s}_Q(n)$, representing a *shifted and repeated* version of $\mathbf{s}(n)$, using the following examples.

Example 1: $\mathbf{s}_1(n)$ is $\mathbf{s}(n)$ itself. ■

Example 2: Consider a sequence of 3×1 vectors $\mathbf{s}(n)$ defined as

$$[\mathbf{s}(0) \quad \mathbf{s}(1) \quad \mathbf{s}(2)] = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

and $\mathbf{s}(n) = 0$ for $n \geq 3$. Then $\mathbf{s}_2(n)$ can be expressed as

$$[\mathbf{s}_2(0) \quad \mathbf{s}_2(1) \quad \mathbf{s}_2(2) \quad \mathbf{s}_2(3) \quad \mathbf{s}_2(4) \quad \mathbf{s}_2(5)] \tag{1}$$

$$= \begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 2 & 1 & 5 & 4 & 8 & 7 \\ 3 & 2 & 6 & 5 & 9 & 8 \\ 0 & 3 & 0 & 6 & 0 & 9 \end{bmatrix} \tag{2}$$

and $\mathbf{s}_2(n) = 0$ for $n \geq 6$. And $\mathbf{s}_3(n)$ can be expressed as

$$[\mathbf{s}_3(0), \mathbf{s}_3(1), \mathbf{s}_3(2), \mathbf{s}_3(3), \mathbf{s}_3(4), \mathbf{s}_3(5), \mathbf{s}_3(6), \mathbf{s}_3(7), \mathbf{s}_3(8)] \tag{3}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 4 & 0 & 0 & 7 & 0 & 0 \\ 2 & 1 & 0 & 5 & 4 & 0 & 8 & 7 & 0 \\ 3 & 2 & 1 & 6 & 5 & 4 & 9 & 8 & 7 \\ 0 & 3 & 2 & 0 & 6 & 5 & 0 & 9 & 8 \\ 0 & 0 & 3 & 0 & 0 & 6 & 0 & 0 & 9 \end{bmatrix} \tag{4}$$

and $\mathbf{s}_3(n) = 0$ for $n \geq 9$. ■

The formal definition of $\mathbf{s}_Q(n)$ is given as follows.

Definition 2: Given a positive integer Q and a sequence of $M \times 1$ vectors $\mathbf{s}(n)$ over the field $\mathbb{C}$, $\mathbf{s}_Q(n)$ is a sequence of $(M+Q-1) \times 1$ vectors defined as

$$\mathbf{s}_Q(nQ+k) = \begin{bmatrix} \mathbf{0}_{k \times 1} \\ \mathbf{s}(n) \\ \mathbf{0}_{(Q-k-1) \times 1} \end{bmatrix}$$

for $n \geq 0$, $k = 0, 1, \ldots, Q-1$. ■

Note that the matrices shown in (2) and (4) are similar to Sylvester's resultant matrices [6]in the manner of Toeplitz-like structures. The definition of generalized signal richness is given as follows.

Definition 3: An $M \times 1$ sequence $\mathbf{s}(n)$, $n \geq 0$, is said to be $(1/Q)$-*rich* if $\mathbf{s}_Q(n)$ is rich. ■

Note that when $Q = 1$, Definition 3 reduces to the conventional signal richness given in Definition 1. For the example given in Example 2, we can verify that $\mathbf{s}(n)$ is (1/2)-rich and (1/3)-rich but not 1-rich.

An alternative definition of $(1/Q)$-richness can be given immediately by using the following theorem.

Theorem 1 ($(1/Q)$-Richness): Given an $M \times 1$ vector sequence $\mathbf{s}(n)$, $n \geq 0$, $\mathbf{s}(n)$ is $(1/Q)$-rich if and only if there does not exist a nonzero $Q \times M$ Hankel matrix $\mathbf{H}$ such that $\mathbf{H}\mathbf{s}(n) = \mathbf{0}$, $\forall n \geq 0$.

Proof: See the Appendix. ■

B. Properties of $(1/Q)$-Richness

It can be shown that the condition of $(1/Q)$-richness is stronger when the integer Q is smaller, as shown in the following lemma.

Lemma 1: If a sequence of $M \times 1$ vectors $\mathbf{s}(n)$, $n \geq 0$ is $(1/Q)$-rich, then $\mathbf{s}(n)$ is $(1/(Q+1))$-rich.

Proof: The proof of this lemma becomes straightforward when we use the result of Theorem 1. Suppose $\mathbf{s}(n)$ is $(1/Q)$-rich but not $(1/(Q+1))$-rich. Then there exists a nonzero $(Q+1) \times M$ Hankel matrix $\mathbf{V}$ such that $\mathbf{V}\mathbf{s}(n) = \mathbf{0}$ for all n. Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be $Q \times M$ Hankel matrices whose rows are composed of the first Q rows of $\mathbf{V}$ and the last Q rows of $\mathbf{V}$, respectively. Note that at least one of $\mathbf{V}_1$ and $\mathbf{V}_2$ is nonzero and $\mathbf{V}\mathbf{s}(n) = \mathbf{0}$ implies $\mathbf{V}_k\mathbf{s}(n) = \mathbf{0}$ for $k = 1, 2$. This violates the assumption that $\mathbf{s}(n)$ is $(1/Q)$-rich. ■

Lemma 1 states a basic property of generalized signal richness: the smaller the value of Q is, the "stronger" the condition of $(1/Q)$-richness is. For example, if an $M \times 1$ sequence $\mathbf{s}(n)$ is 1-rich, or simply *rich*, then it is (1/2)-rich, (1/3)-rich, and $(1/Q)$-rich for any positive integer Q. This is why we use the notation of $(1/Q)$-richness. On the contrary, a (1/2)-rich signal $\mathbf{s}(n)$ is not necessarily 1-rich. We can thus define a *measure* of generalized signal richness, namely the *degree of nonrichness* for a given $M \times 1$ sequence $\mathbf{s}(n)$ as follows.

Definition 4: Given an $M \times 1$ sequence $\mathbf{s}(n)$, $n \geq 0$, the *degree of nonrichness* of $\mathbf{s}(n)$ is defined as

$$Q_{\min} \triangleq \min_{Q} \left(\mathbf{s}(n) \text{ is } \frac{1}{Q} - \text{rich} \right). \tag{5}$$

If $\mathbf{s}(n)$ is not $(1/Q)$-rich for any Q, then $Q_{\min} = \infty$. The property of an infinite degree of nonrichness can be described in the following lemma, in which we use the notation of $\mathbf{p}_M(x)$ to denote the column vector:

$$\mathbf{p}_M(x) = [1 \quad x \quad x^2 \quad \cdots \quad x^{M-1}]^T.$$

Lemma 2: Consider a sequence of $M \times 1$ vectors $\mathbf{s}(n), n \geq 0$. The following statements are equivalent.

1) $\mathbf{s}(n)$ is not $(1/Q)$-rich for any Q.
2) The degree of nonrichness of $\mathbf{s}(n)$ is infinity.
3) Either $\mathbf{p}_M^T(\alpha)\mathbf{s}(n) = 0, \forall n$ for some $\alpha \in \mathbb{C}$ or $[0\ 0\ \cdots\ 1]\mathbf{s}(n) = 0, \forall n$.
4) Polynomials $p_n(x) = \mathbf{p}_M^T(x)\mathbf{s}(n)$, $n \geq 0$, either share a common zero $\alpha \in \mathbb{C}$ or all have orders less than $M-1$.

Proof: See [5]. ■

Lemma 1 suggests that if the value of Q is larger, the less "rich" is the signal $\mathbf{s}(n)$. By definition, a 1-rich signal has "full rank." If $\mathbf{s}(n)$ is not 1-rich but has only one annihilator $\mathbf{v}^T$ (i.e., $\mathbf{v}^T\mathbf{s}(n) = 0$), intuitively it is still likely to be (1/2)-rich, or $(1/Q)$-rich for other larger Q. Lemma 2 suggests, however, this is not the case if the annihilator happens to be in the form defined in condition 3) of Lemma 2. If an $M \times 1$ sequence $\mathbf{s}(n)$ has a finite degree of nonrichness, or $\mathbf{s}(n)$ is $(1/Q)$-rich for some integer Q, then it can be shown that the maximum possible value of $Q_{\min}$ is $M-1$, as described in the following lemma.

Lemma 3: If $M > 1$ and an $M \times 1$ sequence $\mathbf{s}(n)$ is not $(1/(M-1))$-rich, then it is not $(1/Q)$-rich for any Q.

Proof: See [5]. ■

With Lemma 3, we can see that for an $M \times 1$ sequence $\mathbf{s}(n)$, $(1/(M-1))$-richness is the weakest form of generalized richness. Given an $M \times 1$ vector sequence $\mathbf{s}(n)$, the degree of nonrichness can only be one of values $1, 2, \ldots, M-1$, or ∞.

C. Vandermonde-Form Vectors and Generalized Zero Location

Consider a $1 \times M$ complex-valued row vector $\mathbf{v}^T = [v_1\ v_2\ \cdots\ v_M]$ which has the form

$$\mathbf{v}^T = c[1 \quad \alpha \quad \alpha^2 \quad \cdots \quad \alpha^{M-1}] \tag{6}$$

for some $c, \alpha \in \mathbb{C}$, $c \neq 0$. We call a vector in the form of (6) a *Vandermonde-form vector* since it can be a row of a Vandermonde matrix. Now, consider the vector

$$\mathbf{v}^T = [0 \quad 0 \quad \cdots \quad 0 \quad c] \tag{7}$$

for some $c \in \mathbb{C}$, $c \neq 0$. In view of condition 3) of Lemma 2, for generality we want to include vectors as in (7) into the definition of Vandermonde-form vectors. A formal definition of Vandermonde-form vectors is given as follows.

Definition 5 (Vandermonde-Form Vectors): A row vector $\mathbf{v}^T = [v_1\ v_2\ \cdots\ v_M]$ is said to be in the "*Vandermonde form*" if there exist $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 > 0$, such that

$$\mathbf{v}^T = [\beta^{M-1} \quad \alpha\beta^{M-2} \quad \cdots \quad \alpha^{M-2}\beta \quad \alpha^{M-1}].$$

The set of M-vectors in Vandermonde form, denoted as $\mathcal{V}_M$, is defined as

$$\mathcal{V}_M^T = \{\mathbf{v}^T | \mathbf{v} \in \mathbb{C}^M \text{ and } \mathbf{v}^T \text{ is in the Vandermonde form}\}.$$

■

By the definition above, we have

$$\begin{aligned}\mathbf{v}^T &= [\beta^{M-1} \quad \alpha\beta^{M-2} \quad \cdots \quad \alpha^{M-2}\beta \quad \alpha^{M-1}] \\ &= \begin{cases} \beta^{M-1} \cdot \mathbf{p}_M^T\left(\frac{\alpha}{\beta}\right), & \text{if } \beta \neq 0 \\ [0\ \ 0\ \ \cdots\ \ \alpha^{M-1}], & \text{if } \beta = 0. \end{cases}\end{aligned}$$

A straightforward observation on Definition 5 is described below.

Property 1: If $M \leq 2$, a nonzero $M \times 1$ row vector $\mathbf{v}^T$ is always a Vandermonde-form vector.

Proof: Self-evident. ■

In view of Definition 5, it would be useful if we define a *Vandermonde ratio* for each M-row vector in Vandermonde form.

Definition 6 (Vandermonde Ratio): For a row vector $\mathbf{v}^T \in \mathcal{V}_M$

$$\mathbf{v}^T = [\beta^{M-1} \quad \alpha\beta^{M-2} \quad \cdots \quad \alpha^{M-2}\beta \quad \alpha^{M-1}]$$

where $\alpha, \beta \in \mathbb{C}$, the "*Vandermonde ratio*" $\gamma \in \mathbb{C} \bigcup \{\infty\}$ is defined as

$$\gamma = \begin{cases} \frac{\alpha}{\beta}, & \text{if } \beta \neq 0 \\ \infty, & \text{if } \beta = 0. \end{cases}$$

■

Lemma 4: Let $\mathbf{v}^T$ be a $1 \times M$ Vandermonde vector with Vandermonde ratio $\gamma \in \mathbb{C} \bigcup \{\infty\}$. Let $\mathbf{y}$ be an $M \times 1$ nonzero vector. Then $\mathbf{v}^T\mathbf{y} = 0$ if and only if we have the following.

1) Polynomial $\mathbf{p}_M^T(x)\mathbf{y}$ has a zero at γ if $\gamma \in \mathbb{C}$.
2) Polynomial $\mathbf{p}_M^T(x)\mathbf{y}$ has a degree less than $M-1$ if $\gamma = \infty$.

Proof: See the Appendix. ■

Now, let us turn our attention to the sequence of polynomials $p_n(x) = \mathbf{p}_M^T(x)\mathbf{s}(n), n = 0, 1, 2, \ldots$. Lemma 2 states that $\mathbf{s}(n)$ has an infinite degree of nonrichness if and only if the polynomials $p_n(x)$ either a) have a common factor or b) all have an order less than $M-1$. Conditions a) and b), although seemingly unrelated to each other, can be unified in one statement using the following definition.

Definition 7: Given an $M \times 1$ nonzero column vector $\mathbf{u}$. Suppose $u(x) = \mathbf{p}_M^T(x)\mathbf{u}$ is an mth-order polynomial, where $m+1 \leq M$, (i.e., $[\mathbf{u}]_l = 0, \forall l \in \{m+2, m+3, \ldots, M\}$). The "*zero locations*" of $\mathbf{u}$, is defined as a multiset $\mathcal{Z}_\mathbf{u}$ of $M-1$ elements from $\mathbb{C} \bigcup \{\infty\}$ (possibly with multiplicity), as follows:

$$\mathcal{Z}_\mathbf{u} = \{\alpha_1, \alpha_2, \ldots, \alpha_m, \infty, \ldots, \infty\}$$

where $\alpha_1, \ldots, \alpha_m$ are the zeros of the polynomial $\mathbf{p}_M^T(x)\mathbf{u}$ whose degree is m. The number of occurrences of ∞ is $M - m - 1$. ■

Example 3: For example, if $\mathbf{y} = [1\ -2\ 1]^T$, then $\mathcal{Z}_\mathbf{y} = \{1, 1\}$.

If $\mathbf{y} = [1\ -2\ 1\ 0]^T$, then $\mathcal{Z}_\mathbf{y} = \{1, 1, \infty\}$.

If $\mathbf{y} = [1\ -3\ 2]^T$, then $\mathcal{Z}_\mathbf{y} = \{1, 1/2\}$.

If $\mathbf{y} = [1\ -3\ 2\ 0\ 0]^T$, then $\mathcal{Z}_\mathbf{y} = \{1, 1/2, \infty, \infty\}$.

As an extreme case, if $\mathbf{y} = [1\ 0\ 0\ 0]$, then $\mathcal{Z}_\mathbf{y} = \{\infty, \infty, \infty\}$. ■

This definition may seem unusual at the first sight since infinity can never be a zero of a polynomial. Nevertheless, we gave this definition *on a vector* for convenience in our context and will find it useful in later discussions. So far, we have not given

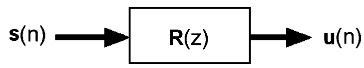

Fig. 1. Multi-input multioutput LTI system.

a formal definition of set of zero locations on a zero vector $\mathbf{0}$. However, there is no loss of generality in the following discussions to assume that

$$\mathcal{Z}_{\mathbf{0}} = \biguplus_{m=1}^{M-1} \left(\mathbb{C} \bigcup \{\infty\} \right)$$

which means any number in the complex plane is a zero location of the vector $\mathbf{0}$ with a multiplicity $M-1$.

With the new definitions addressed above, we can rewrite Lemma 2 in a clearer manner.

Lemma 5 (Lemma 2 Rewritten): Consider a sequence of $M \times 1$ vector $\mathbf{s}(n)$, $n \geq 0$. The following statements are equivalent.

1) $\mathbf{s}(n)$ is not $(1/Q)$-rich for any Q.
2) The degree of nonrichness of $\mathbf{s}(n)$ is infinity.
3) There exists a Vandermonde-form vector $\mathbf{v}^T \in \mathcal{V}_M$ (with a Vandermonde ratio $\gamma \in \mathbb{C} \bigcup \{\infty\}$) such that $\mathbf{v}^T \mathbf{s}(n) = 0$, $\forall n \geq 0$.
4) $\exists \gamma \in \mathbb{C} \bigcup \{\infty\}$ such that $\gamma \in \bigcap_{n=0}^{\infty} \mathcal{Z}_{\mathbf{s}(n)}$ (i.e., vectors $\mathbf{s}(n)$, $n \geq 0$ share a common zero $\gamma \in \mathbb{C} \bigcup \{\infty\}$). ■

Using Lemmas 2 and 3, we readily obtain the following useful lemma.

Lemma 6: Column vectors $\mathbf{s}(n)$, $n \geq 0$ have no common zeros if and only if $\mathbf{s}(n)$ is $(1/(M-1))$-rich.

III. Preserving Generalized Signal Richness

A. Problem Statement

In this section, we will describe the main problem addressed in this paper. Consider an Nth-order, M-input, M-output LTI causal system, depicted in Fig. 1, with a transfer function $\mathbf{R}(z) = \sum_{k=0}^{N} \mathbf{R}_k z^{-k}$.

Definition 8: An $M \times M$ system $\mathbf{R}(z) = \sum_{k=0}^{N} \mathbf{R}_k z^{-k}$ is said to be $(1/Q)$-*richness preserving* if and only if for any $(1/Q)$-rich signal $\mathbf{s}(n)$, the output $\mathbf{u}(n) = \sum_{k=0}^{N} \mathbf{R}_k \mathbf{s}(n-k)$ is also a $(1/Q)$-rich signal. ■

We want to find out the necessary and sufficient conditions for the LTI systems $\mathbf{R}(z)$ to be $(1/Q)$-richness preserving. The special case of this problem when $Q = 1$ was solved in [10]. In particular, for memoryless systems, an $M \times M$ constant matrix $\mathbf{R}$ preserves 1-richness if and only if $\mathbf{R}$ is nonsingular. However, in the case when $Q > 1$, a nonsingular memoryless system $\mathbf{R}$ does not necessarily preserve $(1/Q)$-richness. This can be seen in the following simple example.

Example 4: Let $\mathbf{s}(0) = [1 \ -1 \ 0]^T$, $\mathbf{s}(1) = [1 \ 1 \ 2]^T$, and $\mathbf{s}(n) = \mathbf{0}$ for $n > 1$. By observing that

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

has full rank 4, we know that $\mathbf{s}(n)$ is (1/2)-rich. Now let

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

which is an invertible permutation matrix. Then we can obtain the output $\mathbf{u}(n) = \mathbf{R}\mathbf{s}(n)$ as $\mathbf{u}(0) = [1 \ 0 \ -1]^T$ and $\mathbf{u}(1) = [1 \ 2 \ 1]^T$. Note that if $\mathbf{v}^T = \mathbf{p}_3^T(-1) = [1 \ -1 \ 1]$, then $\mathbf{v}^T \mathbf{u}(n) = 0$ for all n. So $\mathbf{u}(n)$ is not $(1/Q)$-rich for any Q. This suggests that an invertible constant precoder, although preserving the "rank" of a signal, does not preserve $(1/Q)$-richness in general! ■

In this paper we will limit our focus of the problem on memoryless systems, as described below.

Main Problem: Given integers M, Q, where $M > 1$ and $1 \leq Q \leq M-1$, what are the necessary and sufficient conditions for an $M \times M$ matrix $\mathbf{R}$ to be $(1/Q)$-richness preserving?

B. Special Case When $Q = M - 1$

From Lemmas 2, 3, and 6, we know that $\mathbf{s}(n)$ is $(1/(M-1))$-rich if and only if there is no row vector $\mathbf{v}^T \in \mathcal{V}_M$ such that $\mathbf{v}^T \mathbf{s}(n) = 0$, $\forall n \geq 0$. This suggests that a $(1/(M-1))$-richness preserving matrix $\mathbf{R}$ may have something to do with Vandermonde-form vectors.

Theorem 2: An $M \times M$ matrix $\mathbf{R}$ preserves $1/(M-1)$-richness if and only if $\mathbf{v}^T\mathbf{R} \in \mathcal{V}_M$ for all $\mathbf{v}^T \in \mathcal{V}_M$. An $M \times M$ constant matrix $\mathbf{R}$ satisfying this condition is said to be a "*Vandermonde-form preserving*" (VFP) matrix.

Proof: See the Appendix. ■

While a rigorous proof of Theorem 2 can be found in the Appendix, here we seek to present an intuitive understanding of it. Recall that signal $\mathbf{s}(n)$ being $1/(M-1)$-rich means that vectors $\mathbf{s}(n)$, $n \geq 0$ do not share a common zero $\gamma \in \mathbb{C} \bigcup \{\infty\}$ (see Definition 7). Denote the set of zeros of the vector $\mathbf{s}(n)$ as $\mathcal{Z}_{\mathbf{s}(n)}$. Then we have

$$\bigcap_{n=0}^{+\infty} \mathcal{Z}_{\mathbf{s}(n)} = \phi.$$

If the matrix $\mathbf{R}$ is chosen arbitrarily, the zeros of the vector $\mathbf{R}\mathbf{s}(n)$ for a given n, $\mathcal{Z}_{\mathbf{R}\mathbf{s}(n)}$, compared to $\mathcal{Z}_{\mathbf{s}(n)}$, are likely to "reshuffle randomly." This is mainly because the zero locations of a vector are a nonlinear function of the vector contents, so it is usually hard to decide $\mathcal{Z}_{\mathbf{R}\mathbf{s}(n)}$ simply by inspecting $\mathcal{Z}_{\mathbf{s}(n)}$. Hence one usually can manage to find a sequence of vectors $\mathbf{s}(n)$ which do not share common zeros but vectors $\mathbf{R}\mathbf{s}(n)$ do share a common zero. On the other hand, if we choose $\mathbf{R}$ as a VFP matrix defined above, each zero of $\mathbf{R}\mathbf{s}(n)$ can be uniquely "predicted," given the zeros of $\mathbf{s}(n)$: suppose $\alpha \in \mathbb{C} \bigcup \{\infty\}$ is a zero of $\mathbf{s}(n)$, that is, there exists $\mathbf{v}^T \in \mathcal{V}_M$ with Vandermonde ratio α such that $\mathbf{v}^T \mathbf{s}(n) = 0$. Then the Vandermonde ratio of $\mathbf{w}^T = \mathbf{v}^T \mathbf{R}$, say γ, must be a zero location of the vector $\mathbf{R}\mathbf{s}(n)$. As we will show in the Section IV-B, the transformation of zero locations, due to the VFP matrix, is a one-to-one mapping. Thus if the vectors $\mathbf{s}(n)$ do not share a common zero, then vectors $\mathbf{R}\mathbf{s}(n)$ also will not have a common zero.

IV. Vandermonde-Form Preserving Matrices

Given the knowledge that Vandermonde-form preserving (VFP) matrices preserves $(1/(M-1))$-richness, we will consider in this section the representation of general $M \times M$ Vandermonde-form preserving (VFP) matrices. We will also present several properties of VFP matrices which help to answer the problem addressed in the previous section.

A. Representation of Vandermonde-Form Preserving Matrices

We start from focusing on what VFP matrices look like. Obviously the identity matrix $\mathbf{I}_M$ and any nonzero multiple of it are VFP matrices. A permutation matrix, however, is in general not a VFP matrix, such as the one given in Section III-A. So is there any VFP matrix other than a multiple of an identity matrix? First we recognize that a VFP matrix has the following property.

Lemma 7: If an $M \times M$ matrix $\mathbf{R}$ is a Vandermonde-form preserving matrix, then both the first row of $\mathbf{R}$ and the last row of $\mathbf{R}$, $[\mathbf{R}]_1$ and $[\mathbf{R}]_M$, are in $\mathcal{V}_M$.

Proof: See the Appendix. ■

An identity matrix $\mathbf{I}_M$ certainly satisfies this condition since the first row and the last row, $\mathbf{e}_1^T$ and $\mathbf{e}_M^T$, respectively, are in Vandermonde form. Now if we choose the first row and the last row of an $M \times M$ matrix $\mathbf{R}$ as vectors in $\mathcal{V}_M$ other than $\mathbf{e}_1^T$ and $\mathbf{e}_M^T$, will we be able to construct a VFP matrix $\mathbf{R}$? The answer turns out to be yes if we choose the first row and the last row of $\mathbf{R}$ as two Vandermonde-form vectors with different Vandermonde ratios. The following theorem gives the most general characterization of VFP matrices.

Theorem 3: An $M \times M$ matrix $\mathbf{R} = [\mathbf{r}_1 \ \mathbf{r}_2 \ \cdots \ \mathbf{r}_M]$ is Vandermonde-form preserving if and only if there exists a 2×2 invertible matrix

$$\mathbf{R}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$r_k(x) = (a+cx)^{M-k}(b+dx)^{k-1}, \qquad k = 1, 2, \ldots, M$$

where $r_k(x)$ is the polynomial representation of the column vector $\mathbf{r}_k$, i.e., $r_k(x) = \mathbf{p}_M^T(x)\mathbf{r}_k$ (see definition of $\mathbf{p}_M(x)$ in Section II-B). The 2×2 matrix $\mathbf{R}_2$ is called the *characteristic matrix* of the $M \times M$ VFP matrix $\mathbf{R}$.

Proof: See the Appendix. ■

Theorem 3 essentially provides us a construction method of an $M \times M$ VFP matrix using a "seed" 2×2 nonsingular matrix

$$\mathbf{R}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Note that $\mathbf{R}_2$ is always a VFP matrix as long as it is nonsingular (i.e., $ad - bc \neq 0$) since a 1×2 nonzero vector is always in the Vandermonde form. Besides, we can see that any $M \times M$ VFP matrix $\mathbf{R}_M$ can be parameterized by a 2×2 Vandermonde-form preserving matrix. Thus the number of freedoms of $M \times M$ Vandermonde-form preserving matrices is always a constant for any $M > 1$. For convenience, we denote

$$\mathcal{R}_M\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

where $ad - bc \neq 0$, as the $M \times M$ Vandermonde-form preserving matrix generated with polynomials $a + cx$ and $b + dx$, e.g.,

$$\mathcal{R}_3\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix}. \tag{8}$$

Some more numerical examples are presented below for a better "visual" understanding of VFP matrices.

Example 5: If we choose

$$\mathbf{R}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

then

$$\mathbf{R}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we choose

$$\mathbf{R}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

then

$$\mathbf{R}_3 = \begin{bmatrix} 4 & 2 & 1 \\ 4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{R}_4 = \begin{bmatrix} 8 & 4 & 2 & 1 \\ 12 & 4 & 1 & 0 \\ 6 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

■

Example 6: A VFP matrix can also be a full matrix. If we choose

$$\mathbf{R}_2 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

then

$$\mathbf{R}_3 = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix} \qquad \mathbf{R}_4 = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 3 & 5 & 8 & 12 \\ 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

If we choose

$$\mathbf{R}_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

then

$$\mathbf{R}_3 = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 4 \\ 4 & 2 & 1 \end{bmatrix} \qquad \mathbf{R}_4 = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 6 & 9 & 12 & 12 \\ 12 & 12 & 9 & 6 \\ 8 & 4 & 2 & 1 \end{bmatrix}.$$

If we choose

$$\mathbf{R}_2 = \begin{bmatrix} 1 & j \\ j & 1 \end{bmatrix}$$

then

$$\mathbf{R}_3 = \begin{bmatrix} 1 & j & -1 \\ 2j & 0 & 2j \\ -1 & j & 1 \end{bmatrix} \qquad \mathbf{R}_4 = \begin{bmatrix} 1 & j & -1 & -j \\ 3j & -1 & j & -3 \\ -3 & j & -1 & 3j \\ -j & -1 & j & 1 \end{bmatrix}.$$

■

B. Zero-Location Transformation

The key reason that a VFP matrix preserves $1/(M-1)$-richness is that it transforms each zero location of a column vector (see Definition 7) with a transformation function. This function depends only on its characteristic matrix and is independent from any other zeros of the column vector. In this subsection we will explore how VFP matrices transform zero locations of a column vector.

Consider an $M \times 1$ vector $\mathbf{u}$ and the set of zero locations

$$\mathcal{Z}_{\mathbf{u}} = \{\alpha_1, \alpha_2, \ldots, \alpha_{M-1}\}$$

where $\alpha_k \in \mathbb{C}\bigcup\{\infty\}$ for all $k = 1, 2, \ldots, M-1$, as defined in Definition 7. Now, consider an $M \times M$ VFP matrix $\mathbf{R}$ whose characteristic matrix is

$$\mathbf{R}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Suppose the set of zero locations of $\mathbf{y} = \mathbf{R}\mathbf{u}$ is

$$\mathcal{Z}_{\mathbf{y}} = \{\beta_1, \beta_2, \ldots, \beta_{M-1}\}.$$

How can we find each element of $\mathcal{Z}_{\mathbf{y}}$ given its corresponding zero in $\mathcal{Z}_{\mathbf{u}}$ and the values of $\mathbf{R}_2$? This question is directly related to how the Vandermonde ratio of $\mathbf{w}^T$ is related to that of $\mathbf{v}^T$ when $\mathbf{w}^T = \mathbf{v}^T\mathbf{R}$, as presented in the following theorem.

Theorem 4: Suppose $\mathbf{v}^T \in \mathcal{V}_M$ has a Vandermonde ratio $\alpha \in \mathbb{C}\bigcup\{\infty\}$ and $\mathbf{R}_M$ is a VFP matrix with a nonsingular characteristic matrix

$$\mathbf{R}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then $\mathbf{w}^T = \mathbf{v}^T\mathbf{R}_M$ is also a Vandermonde-form vector with Vandermonde ratio $\beta = f(\alpha)$ where $f : \mathbb{C}\bigcup\{\infty\} \to \mathbb{C}\bigcup\{\infty\}$ is called the *characteristic function* of $\mathbf{R}_M$, defined as

$$f(\alpha) = \lim_{x \to \alpha} \frac{b + dx}{a + cx}. \tag{9}$$

Proof: See the Appendix. ■

In view of Theorem 4, when $a + c\alpha = 0$, the function f gives the value of infinity. On the other hand, if α is infinity, the function gives the value d/c when $c \neq 0$ or gives the value ∞ when $c = 0$ and $d \neq 0$. Notice that c and d cannot both be zero due to the nonsingularity of $\mathbf{R}_2$ matrix. Also note that the characteristic function of a VFP matrix depends only on the 2 $\times$ 2 characteristic matrix and not on the size of the VFP matrix. Some numerical examples are presented below to demonstrate Theorem 4 and clarify the concept.

Example 7: We take

$$\mathbf{R}_2 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

as in Example 6. Then the 4×4 VFP matrix characterized by $\mathbf{R}_2$ is

$$\mathbf{R}_4 = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 3 & 5 & 8 & 12 \\ 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The characteristic function of $\mathbf{R}_4$ is

$$f(\alpha) = \lim_{x \to \alpha} \frac{2 + x}{1 + x}.$$

Let $\mathbf{v}^T = [1 \ -3 \ 9 \ -27]$, which has a Vandermonde ratio $\alpha = -3$. Then

$$\mathbf{w}^T = \mathbf{v}^T\mathbf{R}_4 = [-8 \quad -4 \quad -2 \quad -1]$$

has a Vandermonde ratio $\beta = (2-3)/(1-3) = 1/2$.

If $\mathbf{v}^T = [1 \ -1 \ 1 \ -1]$, which has a Vandermonde ratio $\alpha = -1$, then

$$\mathbf{w}^T = \mathbf{v}^T\mathbf{R}_4 = [0 \quad 0 \quad 0 \quad 1]$$

has a Vandermonde ratio $\beta = \infty$.

If $\mathbf{v}^T = [0 \ 0 \ 0 \ 1]$, which has a Vandermonde ratio $\alpha = \infty$, then

$$\mathbf{w}^T = \mathbf{v}^T\mathbf{R}_4 = [1 \quad 1 \quad 1 \quad 1]$$

has a Vandermonde ratio $\beta = 1/1 = 1$. ■

From the discussions above, we find that a VFP matrix "bilinearly" transforms the Vandermonde ratio of a Vandermonde-form vector with the characteristic function f defined in Theorem 4. Note that the function f is a one-to-one and onto function. The inverse function of f can be expressed as

$$g(\beta) = \lim_{y \to \beta} \left(-\frac{ay - b}{cy - d}\right). \tag{10}$$

A direct corollary of Theorem 4 is presented below.

Corollary 1: If β is a zero with multiplicity m of an $M \times 1$ vector $\mathbf{u}$, then

$$\alpha = g(\beta) = \lim_{y \to \beta} \left(-\frac{ay - b}{cy - d}\right)$$

is a zero with multiplicity m of the vector $\mathbf{y} = \mathbf{R}_M\mathbf{u}$.

Proof: Since $\beta \in \mathcal{Z}_{\mathbf{u}}$, we have $\mathbf{v}_\beta^T\mathbf{u} = \mathbf{0}$ where $\mathbf{v}_\beta^T \in \mathcal{V}_M$ whose Vandermonde ratio is β. From Theorem 4 there exists $\mathbf{v}_\alpha^T \in \mathcal{V}_M$ whose Vandermonde ratio is $\alpha = f^{-1}(\beta) = g(\beta)$ such that $\mathbf{v}_\beta^T = \mathbf{v}_\alpha^T\mathbf{R}_M$. Then $\mathbf{v}_\alpha^T\mathbf{y} = \mathbf{v}_\alpha^T\mathbf{R}_M\mathbf{u} = \mathbf{v}_\beta^T\mathbf{u} = \mathbf{0}$. So $\alpha = g(\beta) \in \mathcal{Z}_{\mathbf{y}}$. ■

Example 8: We choose the same $\mathbf{R}_4$ as in Example 7. Let $\mathbf{u} = [1 \ -3 \ 2 \ 0]^T$, which has zeros at $\beta_1 = 1/2$, $\beta_2 = \infty$, and $\beta_3 = 1$, respectively. Then we have

$$\begin{aligned} \mathbf{y} &= \mathbf{R}_4\mathbf{u} \\ &= \begin{bmatrix} 1 & 2 & 4 & 8 \\ 3 & 5 & 8 & 12 \\ 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}. \end{aligned}$$

The zero locations of $\mathbf{y}$ are at $\alpha_1 = -3$, $\alpha_2 = -1$, and $\alpha_3 = \infty$. Note that α_k and β_k have the relationship as predicted in Corollary 1. The function g defined in (10) is thus called the *zero-location transformation* (ZLT) function of the VFP matrix $\mathbf{R}_M$. ■

C. Other Properties of VFP Matrices

Some other noteworthy properties of VFP matrices, although not directly related to solving the main problem, are briefly presented here. The reader can verify these with some effort.

1) First of all, VFP matrices are in general not Hermitian nor symmetric, even if the 2×2 characteristic matrix is. In fact, one can prove that for $M > 2$, if $M \times M$ matrix $\mathbf{R}$ is both VFP and Hermitian, then $\mathbf{R}$ must be a diagonal matrix or an antidiagonal matrix (i.e., $[\mathbf{R}]_{ij}$ could be nonzero only when $i + j = M + 1$).
2) VFP matrices are invertible. The inverse of a VFP matrix is also a VFP matrix. In addition, the characteristic function of the inverse of a VFP matrix [as defined in (9)] is the inverse function [as defined in (10)] of the characteristic function of the original VFP matrix.
3) The product of two VFP matrices is a VFP matrix. The characteristic function of the product is the composition of two characteristic functions of the original two VFP matrices.
4) DFT and IDFT matrices are in general not VFP unless $M = 2$. It can also be shown that Hadamard matrices are not VFP in general. This means some most commonly used precoders do not preserve $1/(M-1)$-richness. It can also be shown that a unitary matrix is not VFP unless it is the identity matrix (or a nonzero scaled version of it) or an antidiagonal matrix with identical antidiagonal entries.
5) Define the set of all characteristic functions
$$\mathcal{T}_{\mathbb{C}} = \Big\{ f : \mathbb{C}\bigcup\{\infty\} \to \mathbb{C}\bigcup\{\infty\} | \quad f(\alpha) = \lim_{x\to\alpha} \frac{b+dx}{a+cx}, a,b,c,d \in \mathbb{C}, ad \neq bc \Big\}.$$
Then $(\mathcal{T}_{\mathbb{C}}, \circ)$, where "$\circ$" denotes the function composition operation, is a group which is algebraically isomorphic to the group $(\mathcal{R}_M, \cdot)$, where $\mathcal{R}_M$ is the set of all $M \times M$ VFP matrices and "$\cdot$" is the matrix multiplication operation.
6) Eigenvalues and eigenvectors of a VFP matrix can be easily found given its size M and its 2×2 characteristic matrix. Suppose $\mathbf{R}_M$ is a VFP matrix with a characteristic matrix
$$\mathbf{R}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
whose eigenvalues are λ_1 and λ_2. Then the M eigenvalues of $\mathbf{R}_M$ are
$$\{\lambda_1^{M-1}, \lambda_1^{M-2}\lambda_2, \ldots, \lambda_2^{M-1}\}.$$
So the determinant of $\mathbf{R}_M$ is
$$\det(\mathbf{R}_M) = (\lambda_1\lambda_2)^{M(M-1)/2} = \det(\mathbf{R}_2)^{M(M-1)/2}.$$
Now suppose $\mathbf{u}_i$ is an eigenvector of $\mathbf{R}_2$ associated with the eigenvalue λ_i for $i = 1, 2$. That is, $\mathbf{R}_2 = \mathbf{U}_2\mathbf{\Lambda}_2\mathbf{U}_2^{-1}$ where $\mathbf{U}_2 = [\mathbf{u}_1 \ \mathbf{u}_2]$ and $\mathbf{\Lambda}_2 = \mathrm{diag}(\lambda_1, \lambda_2)$. Then it can be shown that $\mathbf{R}_M = \mathbf{U}_M\mathbf{\Lambda}_M\mathbf{U}_M^{-1}$ where $\mathbf{U}_M = \mathcal{R}_M(\mathbf{U}_2)$ [see definition in (8)] and $\mathbf{\Lambda}_M = \mathrm{diag}(\lambda_1^{M-1}, \lambda_1^{M-2}\lambda_2, \ldots, \lambda_2^{M-1})$.
7) Using the property mentioned above, a VFP matrix with unit-norm eigenvalues can be easily constructed by simply choosing a characteristic matrix $\mathbf{R}_2$ whose eigenvalues λ_1 and λ_2 satisfy $|\lambda_1| = |\lambda_2| = 1$. But it should be noticed that matrices created in this way are usually still not unitary. In fact, one can show that for $M > 2$, an $M \times M$ VFP matrix $\mathbf{R}$ is in general not a normal matrix (i.e., $\mathbf{R}^\dagger\mathbf{R} = \mathbf{R}\mathbf{R}^\dagger$) [7]unless $\mathbf{R}$ is diagonal or antidiagonal. This more general fact also explains properties 1) and 4) mentioned above.

D. VFP Matrices as a Linear Precoder

In real applications when a VFP matrix is used as the precoder, we multiply the input vectors by the VFP matrix at the transmitter and multiply the inverse of the VFP matrix (which is also a VFP matrix) at the receiver after equalization. In many applications, we may want to choose an optimal VFP matrix that satisfies certain constraints (e.g. power constraint, noise reduction, etc.). Since all VFP matrices can be characterized using four parameters (see Theorem 3), an optimization problem can be formulated with respect to only four parameters according to the specific application. In addition, since a VFP matrix is in general not unitary as discussed above, at the receiver it can amplify the signal subspace and noise subspace with different values. Hence, if the channel state information is known to both the transmitter and the receiver, we can accordingly choose the optimal values of $\mathbf{R}_2$ such that the signal-to-noise-ratio (SNR) is maximized.

V. Main Theorem

Now let us return to the problem stated in Section III: what is the necessary and sufficient condition for an $M \times M$ matrix $\mathbf{R}$ to be $(1/Q)$-richness preserving for any Q, $1 \leq Q \leq M-1$? In Section III we have already shown that when $Q = 1$, $\mathbf{R}$ needs to be nonsingular and when $Q = M-1$, $\mathbf{R}$ needs to be Vandermonde-form preserving. With properties of VFP matrices presented in the previous section, we are now ready to solve the general case of problem for any Q, $1 \leq Q \leq M-1$.

A. Necessary Conditions

We first show that the VFP condition is necessary for an $M \times M$ matrix $\mathbf{R}$ to preserve $(1/Q)$-richness for any $Q \geq 2$. From Lemma 2, we learn that if some $\mathbf{v}^T \in \mathcal{V}_M$ is an annihilator of $\mathbf{s}(n)$, then $\mathbf{s}(n)$ cannot be $(1/Q)$-rich for any Q. On the other hand, if some $\mathbf{v}^T$ not in $\mathcal{V}_M$ is the only annihilator of $\mathbf{s}(n)$ (i.e., the signal space has rank deficiency equal to one), we can show that the degree of nonrichness of $\mathbf{s}(n)$ is 2. Following this argument, we can easily obtain the following lemma.

Lemma 8: For $M > 1$, consider an $M \times M$ matrix $\mathbf{R}$. If $2 \leq Q \leq M-1$ and $\mathbf{R}$ is $(1/Q)$-richness preserving, then $\mathbf{R}$ must be VFP.

Proof: See the Appendix. ■

Notice that when $Q = 1$, $\mathbf{R}$ is not necessarily VPF to be $(1/Q)$-richness preserving (nonsingularity is sufficient). Lemma 8 is true only when $Q \geq 2$.

B. Hankel-Form Preservation

As for sufficient conditions of the main problem (for the case $Q \geq 2$), we explore in this subsection another property of VFP matrices.

Theorem 5 (Hankel-Form Preservation): Given an $m \times n$ nonzero Hankel matrix $\mathbf{H} = [h_{ij}]$. Let $\mathbf{R}_2$ be a 2×2 invertible matrix. Let $\mathbf{R}_m = \mathcal{R}_m(\mathbf{R}_2)$ and $\mathbf{R}_n = \mathcal{R}_n(\mathbf{R}_2)$ be $m \times m$ and $n \times n$ VFP matrices, respectively (the notation $\mathcal{R}_M(\cdot)$ was defined in Section IV-A). Then $\mathbf{H}' = \mathbf{R}_m^T \mathbf{H} \mathbf{R}_n$ is also a nonzero Hankel matrix.

Proof: See the Appendix. ■

Theorem 5 shows another capability of VFP matrices: besides preserving Vandermonde-form vectors, they also preserve the property of Hankel matrices if we use two VFP matrices with the same characteristic matrix. An example is shown below.

Example 9: Let

$$\mathbf{R}_2 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$
$$\mathbf{R}_3 = \mathcal{R}_3(\mathbf{R}_2)$$

and

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_2 & h_3 & h_4 \end{bmatrix}$$

is a nonzero Hankel matrix. Then we have

$$\begin{aligned} \mathbf{H}' &= \mathbf{R}_2^T \mathbf{H} \mathbf{R}_3 \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} h_1 & h_2 & h_3 \\ h_2 & h_3 & h_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 4 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} h_1+6h_2+12h_3+8h_4 & h_1+4h_2+4h_3 & h_1+2h_2 \\ h_1+4h_2+4h_3 & h_1+2h_2 & h_1 \end{bmatrix} \end{aligned}$$

which is also a nonzero Hankel matrix. ■

C. Main Theorem

Using Theorem 5 and Lemma 8, the main problem described in Section III can now be completely answered by the following theorem.

Theorem 6: ($1/Q$-richness Preservation) For $M > 1$, $2 \leq Q \leq M-1$, an $M \times M$ matrix $\mathbf{R}_M$ is $(1/Q)$-richness preserving if and only if $\mathbf{R}_M$ is Vandermonde-form preserving.

Proof: The necessity comes directly from Lemma 8. As for sufficiency, suppose a Vandermonde-form preserving matrix

$$\mathbf{R}_M = \mathcal{R}_M(\mathbf{R}_2)$$

is not $(1/Q)$-richness preserving for some $Q \geq 2$, where $\mathbf{R}_2$ is a 2×2 invertible matrix. Then there exists a $(1/Q)$-rich signal $\mathbf{s}(n)$ such that the output $\mathbf{y}(n) = \mathbf{R}_M \mathbf{s}(n)$ is not $(1/Q)$-rich. Using Theorem 1, there exists a $Q \times M$ nonzero Hankel matrix

$$\mathbf{V} = \begin{bmatrix} v_1 & v_2 & \cdots & v_M \\ v_2 & v_3 & \cdots & v_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ v_Q & v_{Q+1} & \cdots & v_{Q+M-1} \end{bmatrix}$$

such that $\mathbf{V}\mathbf{y}(n) = \mathbf{0}$ for all $n \geq 0$. This implies $\mathbf{V}\mathbf{R}_M \mathbf{s}(n) = \mathbf{0}$ for all $n \geq 0$. Let

$$\mathbf{R}_Q = \mathcal{R}_Q(\mathbf{R}_2).$$

We have $\mathbf{R}_Q^T \mathbf{V} \mathbf{R}_M \mathbf{s}(n) = \mathbf{0}$ for all $n \geq 0$. Using Theorem 5, we know that $\mathbf{R}_Q^T \mathbf{V} \mathbf{R}_M$ is also a Hankel matrix. Now using Theorem 1 again, we conclude that $\mathbf{s}(n)$ is also not $(1/Q)$-rich, contradicting the assumption that it is $(1/Q)$-rich. So a Vandermonde-form preserving matrix must be $(1/Q)$-richness preserving for $Q \geq 2$. ■

A summary of the answer of the main problem is given as follows. Given an $M \times M$ matrix $\mathbf{R}$, then

1) when $Q = 1$, $\mathbf{R}$ preserves $(1/Q)$-richness if and only if $\mathbf{R}$ is nonsingular;
2) when $2 \leq Q \leq M-1$, $\mathbf{R}$ preserves $(1/Q)$-richness if and only if $\mathbf{R}$ is a VFP matrix.

VI. Other Relevant Issues on $(1/Q)$-Richness

In this section we will discuss some deeper issues on $(1/Q)$-richness.

A. Relationship Between Degree of Richness and Rank of a Signal

As we already know, given an M-vector signal, the degree of nonrichness $Q_{\min}$ of the signal can only be one of the values $1, 2, \ldots, M-1$, and ∞. And the larger $Q_{\min}$ is, the "less rich" the signal is. And by definition, a signal is 1-rich if and only if a matrix composed of finite sample vectors of $\mathbf{s}(n)$ has full rank M. This gives us an intuition that as the degree of nonrichness of $\mathbf{s}(n)$ increases, the "rank" of $\mathbf{s}(n)$ should decrease. Before further discussion, we shall give a formal definition to the *rank* of a signal $\mathbf{s}(n)$ as follows.

Definition 9: The *rank* of an $M \times 1$ sequence $\mathbf{s}(n)$ is defined as

$$\text{rank}(\mathbf{s}(n)) \triangleq \max_{0 \leq n_1 \leq n_2 \leq \cdots \leq n_M} \text{rank}([\mathbf{s}(n_1), \mathbf{s}(n_2), \ldots, \mathbf{s}(n_M)]).$$

■

In other words, the rank of $\mathbf{s}(n)$ is the maximum number of linearly independent column vectors among $\mathbf{s}(n)$, $n \geq 0$. The rank of an $M \times 1$ signal $\mathbf{s}(n)$ is an integer between zero and M. In particular, if $\text{rank}(\mathbf{s}(n)) = M$, then $Q_{\min} = 1$. If $\text{rank}(\mathbf{s}(n)) \leq 1$, then $Q_{\min} = \infty$. If $\text{rank}(\mathbf{s}(n)) = M-1$, then the degree of nonrichness can be found in the following lemma, which we have already known when exploring necessary conditions of the main problem (see Section V-A).

Lemma 9: If a sequence of $M \times 1$ vectors $\mathbf{s}(n)$ is not 1-rich but $\text{rank}(\mathbf{s}(n)) = M-1$, then the degree of nonrichness of $\mathbf{s}(n)$ is either 2 or ∞.

Proof: See the Appendix. ■

While a high rank signal (as high as $M-1$) can have a "bad" degree of nonrichness as depicted in Lemma 9, a signal with a low degree of nonrichness always implies it has a sufficiently high rank, as explained in the following lemma.

Lemma 10: If an $M \times 1$ sequence $\mathbf{s}(n)$ has a finite degree of nonrichness $Q_{\min}$, then

$$\text{rank}(\mathbf{s}(n)) \geq \frac{M + Q_{\min} - 1}{Q_{\min}}.$$

In particular, if $\text{rank}(\mathbf{s}(n)) = 2$, then $Q_{\min} = M-1$.

Proof: See the Appendix. ■

If $\mathbf{s}(n)$ has a degree of nonrichness $Q_{\min} = 1$, Lemma 10 says the obvious fact that $\text{rank}(\mathbf{s}(n)) = M$. If $Q_{\min} = 2$, then the minimum rank $\mathbf{s}(n)$ must have is $(M+1)/2$. As $Q_{\min}$ increases, the minimum rank required by $\mathbf{s}(n)$ is approximately inverse proportional to $Q_{\min}$, around $1/Q_{\min}$ of full rank. This is also a reason why we call $\mathbf{s}(n)$ $(1/Q)$-rich.

Now let us look at Lemma 10 from the view point of the rank of $\mathbf{s}(n)$. If we consider a signal $\mathbf{s}(n)$ with $\text{rank}(\mathbf{s}(n)) = 2$, then Lemma 10 says $Q_{\min} \geq M-1$. In other words, the degree of nonrichness of $\mathbf{s}(n)$ is either $M-1$ or infinity. More generally, consider an FIR signal with $\mathbf{s}(n) = \mathbf{0}, \forall n \geq J$, i.e., considering an $M \times J$ matrix

$$\mathbf{S} = [\mathbf{s}(0) \quad \mathbf{s}(1) \quad \cdots \quad \mathbf{s}(J-1)].$$

Then the condition in Lemma 10 can be rewritten as

$$J \geq \text{rank}(\mathbf{S}) = \text{rank}\left(\mathbf{s}(n)\right) \geq \frac{M+Q_{\min}-1}{Q_{\min}}. \tag{11}$$

This implies

$$Q_{\min} \geq \left\lceil \frac{M-1}{\text{rank}(\mathbf{S})-1} \right\rceil \geq \left\lceil \frac{M-1}{J-1} \right\rceil. \tag{12}$$

The equality in the left part of inequality (12) always holds true when $M \leq 4$, as long as $\mathbf{s}(n)$ has no annihilator in the Vandermonde form (i.e., $Q_{\min} = \infty$). This can be readily verified using Lemmas 9 and 10. When $M \geq 5$, there are, however, situations when this is not true, as can be seen in the following example.

Example 10: Let $M = 5$ and $\mathbf{s}(n)$ be chosen as

$$[\mathbf{s}(0) \quad \mathbf{s}(1) \quad \mathbf{s}(2)] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $\mathbf{s}(n) = \mathbf{0}$ when $n \geq 3$.

Then $\text{rank}(\mathbf{s}(n)) = 3$. So $Q_{\min} \geq \lceil (5-1)/(3-1) \rceil = 2$ as indicated in (12), and vectors $\mathbf{s}(n)$, $n \geq 0$ do not share a common zero ($\mathcal{Z}_{\mathbf{s}(0)} = \{1.7221, -0.6514 \pm 0.7587j, 0.5807\}$, $\mathcal{Z}_{\mathbf{s}(1)} = \{1.9052, -0.7881 \pm 0.4014j, 0.6710\}$, and $\mathcal{Z}_{\mathbf{s}(2)} = \{0, 0, 1, \infty\}$) so $Q_{\min}$ is finite. However, it can be verified that $\mathbf{s}(n)$ has two annihilators [2 1 1 1 1] and [1 1 1 1 2] and so the 2×5 Hankel matrix

$$\mathbf{H} = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

satisfies $\mathbf{H}\mathbf{s}(n) = \mathbf{0}$. So $\mathbf{s}(n)$ is not (1/2)-rich and $Q_{\min} > 2$ (actually $Q_{\min} = 3$ since the Hankel matrix $\mathbf{H}$ cannot be extended into three rows in this case). ■

We summarize the relationship between degree of nonrichness and rank of an $M \times 1$ sequence $\mathbf{s}(n)$ in Table I.

B. Distribution of Degree of Nonrichness

In this subsection we want to discuss the distribution of degree of nonrichness for a sequence of $M \times 1$ vectors $\mathbf{s}(n)$ when all entries of $\mathbf{s}(n)$ come from a finite constellation. We perform a Monte Carlo experiment with 2 500 000 samples of $8 \times J$ matrices for each J, $2 \leq J \leq 9$, whose entries are randomly chosen from commonly used communication constellations: BPSK, QPSK, and 16-QAM. BPSK constellation has an alphabet size of two (1 and -1). QPSK constellation has a size of four and 16-QAM has a size of 16. Each $8 \times J$ matrix can represent a causal FIR eight-vector signal whose first J samples are nonzero. In Figs. 2–4, the length of each bar segment with specific color represents the proportion of samples which have the corresponding degree of nonrichness $Q_{\min}$. For example, in Fig. 2, around 77% of samples of 8×9 matrices have a degree of nonrichness $Q_{\min} = 1$ while most of the rest have around 23%. In view of these figures, we find that the degree of nonrichness tends to achieve the lower bound predicted in (12) when entries of the signal come from a larger constellation. This indicates that in real applications (see [5] for more detailed reference) where Q is given, it is usually sufficient to collect

$$J = \frac{M+Q-1}{Q}$$

TABLE I
RELATIONSHIP BETWEEN DEGREE OF NONRICHNESS AND RANK OF $\mathbf{s}(n)$. NOTICE AMBIGUITY OF FINITE VALUES FOR $M \geq 5$. SEE TEXT

	Q_{min}				
rank(s(n))	M=2	M=3	M=4	M=5	M=6
1	∞	∞	∞	∞	∞
2	1	2 or ∞	3 or ∞	4 or ∞	5 or ∞
3	-	1	2 or ∞	2, 3, 4, or ∞	3, 4, 5, or ∞
4	-	-	1	2 or ∞	2, 3, 4, 5, or ∞
5	-	-	-	1	2 or ∞
6	-	-	-	-	1

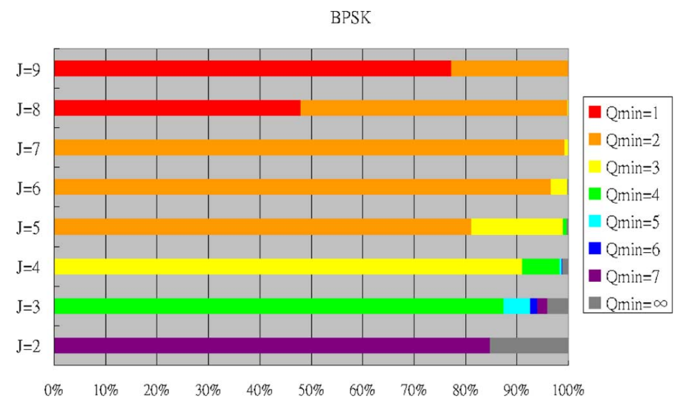

Fig. 2. Distribution of degree of nonrichness of signals whose entries are from BPSK constellation.

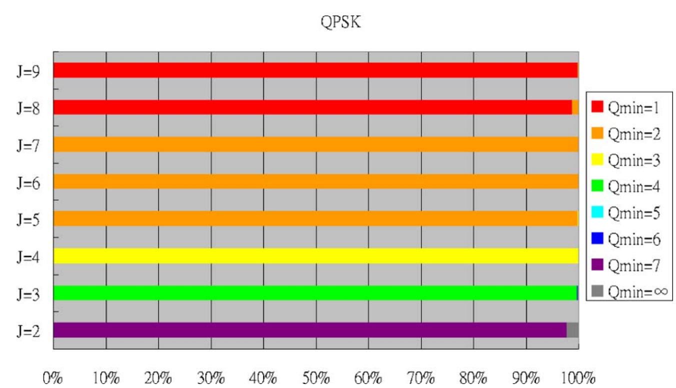

Fig. 3. Distribution of degree of nonrichness of signals whose entries are from the QPSK constellation.

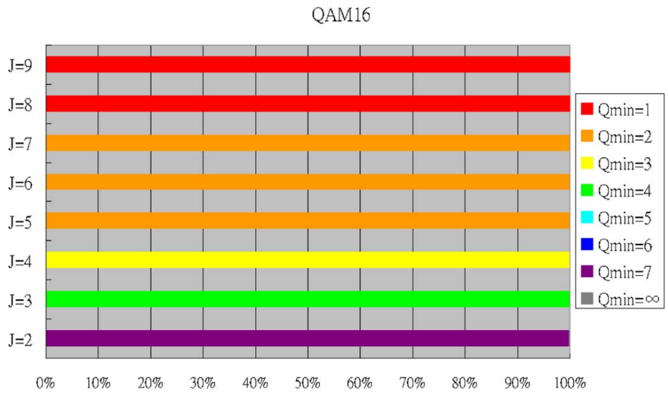

Fig. 4. Distribution of degree of nonrichness of signals whose entries are from the 16-QAM constellation.

samples of vectors when a large constellation is used. On the contrary, when using a small constellation like BPSK, it is quite probable that the signal has a degree of nonrichness larger than the lower bound described in (12).

As a final comment, in real applications when these signals are precoded by a VFP matrix, the degree of nonrichness of the input signal is guaranteed not to decrease. However, since a VFP matrix would not turn a non-$(1/Q)$-rich signal into $(1/Q)$-rich, the degree of nonrichness would not increase and thus would be always unchanged. On the contrary, for an arbitrary non-VFP matrix, although the property of $(1/Q)$-richness could sometimes be destroyed, it is sometimes possible that a non-VFP matrix turns a non-$(1/Q)$-rich signal into a $(1/Q)$-rich signal. Whether an arbitrary matrix increases or decreases the probability of $(1/Q)$-richness is not clear at the time of writing this paper.

VII. Concluding Remarks

In this paper, we described a mathematical problem that arises in some applications on blind channel identification. We introduced Vandermonde-form preserving matrices as a new subclass of invertible matrices which are highly relevant to the problem. Several properties of VFP matrices have been presented clearly and the proof of the answer to the problem has been presented systematically.

In the future, it may be useful to consider the problem in general for a system with memory. That is, the transfer function of the precoder is an $M \times M$ polynomial matrix $\mathbf{R}(z) = \sum_{k=0}^{N} \mathbf{r}(k) z^{-k}$. It is also of interest to deal with a rectangular $P \times M$ system $\mathbf{R}(z)$. Finding other engineering applications of VFP matrices will also be interesting.

Appendix

Proof of Theorem 1: If $\mathbf{s}(n)$ is not $(1/Q)$-rich, there exists a nonzero row vector

$$\mathbf{v}^T = [v_1 \quad v_2 \quad \cdots \quad v_{Q+M-1}]$$

such that $\mathbf{v}^T \mathbf{s}_Q(n) = 0, \forall n \geq 0$. Then we have

$$[v_k \quad v_{k+1} \quad \cdots \quad v_{k+M-1}] \mathbf{s}(n) = 0$$

for all k, $1 \leq k \leq Q$. This leads to

$$\underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_M \\ v_2 & v_3 & \cdots & v_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ v_Q & v_{Q+1} & \cdots & v_{Q+M-1} \end{bmatrix}}_{\mathbf{V}} \mathbf{s}(n) = \mathbf{0}$$

for all $n \geq 0$ where $\mathbf{V}$ is a nonzero Hankel matrix.

On the other hand, suppose there exists a nonzero $Q \times M$ Hankel matrix

$$\mathbf{V} = \begin{bmatrix} v_1 & v_2 & \cdots & v_M \\ v_2 & v_3 & \cdots & v_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ v_Q & v_{Q+1} & \cdots & v_{Q+M-1} \end{bmatrix}$$

such that $\mathbf{V}\mathbf{s}(n) = \mathbf{0}$ for all $n \geq 0$. It can be readily verified that the nonzero row vector

$$\mathbf{v}^T = [v_1 \quad v_2 \quad \cdots \quad v_{Q+M-1}]$$

satisfies $\mathbf{v}^T \mathbf{s}_Q(n) = 0$. So $\mathbf{s}(n)$ is not $(1/Q)$-rich. ■

Proof of Lemma 4: If $\gamma \in \mathbb{C}$, the statement is self-evident. If $\gamma = \infty$, then $\mathbf{v}^T = c\mathbf{e}_M^T$ for some $c \neq 0$. So $\mathbf{v}^T \mathbf{y} = 0 \Leftrightarrow \mathbf{e}_M^T \mathbf{y} = 0 \Leftrightarrow [\mathbf{y}]_M = 0 \Leftrightarrow$ polynomial $\mathbf{p}_M^T(x)\mathbf{y}$ does not have the term of x^{M-1} and hence has a degree less than $M-1$. ■

Proof of Theorem 2: Suppose $\mathbf{R}$ is Vandermonde-form preserving (VFP) but does not preserve $(1/(M-1))$-richness (i.e., there exists $\mathbf{s}(n)$ such that $\mathbf{s}(n)$ is $(1/(M-1))$-rich but $\mathbf{R}\mathbf{s}(n)$ is not). Then there exists $\mathbf{w}^T \in \mathcal{V}_M$ such that $\mathbf{w}^T \mathbf{R}\mathbf{s}(n) = 0$. This leads to $\mathbf{v}^T \mathbf{s}(n) = 0$, where $\mathbf{v}^T = \mathbf{w}^T \mathbf{R}$ is also in $\mathcal{V}_M$. This contradicts the fact that $\mathbf{s}(n)$ is $(1/(M-1))$-rich. So $\mathbf{R}$ being VFP implies it preserves $(1/(M-1))$-richness.

On the other hand, if $\mathbf{R}$ is not VFP, then there exists $\mathbf{w}^T \in \mathcal{V}_M$ such that $\mathbf{v}^T = \mathbf{w}^T \mathbf{R}$ is not in $\mathcal{V}_M$. We can thus create a $(1/(M-1))$-rich signal $\mathbf{s}(n)$ such that $\mathbf{v}^T \mathbf{s}(n) = 0, \forall n \geq 0$. (In fact, we can even create a (1/2)-rich signal $\mathbf{s}(n)$, which is stronger than a $(1/(M-1))$-rich signal. See also the proof of Lemma 8.) This implies $\mathbf{w}^T \mathbf{R}\mathbf{s}(n) = 0, \forall n \geq 0$, which means $\mathbf{R}\mathbf{s}(n)$ is not $(1/(M-1))$-rich. So $\mathbf{R}$ does not preserve $(1/(M-1))$-richness. ■

Proof of Lemma 7: We first learn that both $\mathbf{e}_1^T$ and $\mathbf{e}_M^T$ are in $\mathcal{V}_M$ (with Vandermonde ratios 0 and ∞, respectively). Since $[\mathbf{R}]_1 = \mathbf{e}_1^T \mathbf{R}$, $[\mathbf{R}]_M = \mathbf{e}_M^T \mathbf{R}$, and $\mathbf{R}$ is Vandermonde-form preserving, the lemma is proved immediately. ■

Proof of Theorem 3: Let $r_k(x)$ be the polynomial representation of the kth column of $\mathbf{R}$, i.e.,

$$\mathbf{p}_M^T(x)\mathbf{R} = [r_1(x) \quad r_2(x) \quad \cdots \quad r_M(x)].$$

Then we have

$$r_k(x) r_{k+2}(x) = r_{k+1}(x)^2 \tag{13}$$

for $k = 1, 2, \ldots, M-2$. (Otherwise we can find $\gamma \in \mathbb{C}$ such that $r_k(\gamma) r_{k+2}(\gamma) \neq r_{k+1}(\gamma)^2$ and hence $\mathbf{p}_M^T(\gamma)\mathbf{R} \notin \mathcal{V}_M$ while $\mathbf{p}_M^T(\gamma) \in \mathcal{V}_M$.)

We first argue that all columns of $\mathbf{R}$ must be nonzero. If $\mathbf{r}_k = \mathbf{0}$ for some k, then (13) implies that only $\mathbf{r}_1$ and $\mathbf{r}_M$ can be nonzero among $\mathbf{r}_k$'s. If only one of them is nonzero, say $\mathbf{r}_1 \neq \mathbf{0}$ and $\mathbf{r}_M = \mathbf{0}$, then there exists $\mathbf{v}^T \in \mathcal{V}_M$ such that $\mathbf{v}^T\mathbf{r}_1 = 0$ and hence $\mathbf{v}^T\mathbf{R} = \mathbf{0}^T \notin \mathcal{V}_M$. If both $\mathbf{r}_1$ and $\mathbf{r}_M$ are nonzero (which implies $M \geq 3$), then there exists $\gamma \in \mathbb{C}$ such that $r_1(\gamma)$ and $r_M(\gamma)$ are both nonzero. Choose $\mathbf{v}^T = \mathbf{p}_M^T(\gamma) \in \mathcal{V}_M$, then $p_M^T(\gamma)\mathbf{R} = [r_1(\gamma)\ 0\ \cdots\ 0\ r_M(\gamma)] \notin \mathcal{V}_M$.

Since all columns of $\mathbf{R}$ are nonzero, (13) implies that there exist nonzero polynomials $p(x)$ and $q(x)$, which are coprime to each other, such that

$$\frac{r_{k+2}(x)}{r_{k+1}(x)} = \frac{r_{k+1}(x)}{r_k(x)} = \frac{q(x)}{p(x)}$$

for $k = 1, 2, \ldots, M-2$. This leads to

$$r_M(x) = \frac{r_1(x)}{(p(x))^{M-1}} \cdot (q(x))^{M-1}.$$

Since $p(x)$ and $q(x)$ are coprime to each other, we obtain that $(p(x))^{M-1}$ is a factor of $r_1(x)$. So

$$r_1(x) = c(x)\,(p(x))^{M-1}$$

for some nonzero polynomial $c(x)$. We now have

$$r_k(x) = c(x)\,(p(x))^{M-k}\,(q(x))^{k-1}, \qquad k = 1, 2, \ldots, M.$$

Note that $\deg(p(x)) \leq 1$ since otherwise $\deg(r_1(x)) \geq 2(M-1) > M-1$. Similarly we have $\deg(q(x)) \leq 1$. $p(x)$ and $q(x)$ cannot both be constants since otherwise there exists $\mathbf{v}^T \in \mathcal{V}_M$ such that $\mathbf{v}^T\mathbf{R} = \mathbf{0}^T \notin \mathcal{V}_M$. (This $\mathbf{v}^T$ can be chosen as $\mathbf{p}_M^T(\gamma)$ if γ is a zero of $c(x)$. If $c(x)$ is a constant, we can choose $\mathbf{v}^T$ as $[0\ \cdots\ 0\ 1]$.)

Now that at least one of $p(x)$ and $q(x)$ must be a first-order polynomial, $c(x)$ must be a constant, for otherwise either $\deg(r_1(x))$ or $\deg(r_M(x))$ would be greater than $M-1$. Without loss of generality, we can assume $c(x) = 1$. Now let $p(x) = a + cx$ and $q(x) = b + dx$. Since $p(x)$ and $q(x)$ are coprime to each other and they cannot be constants simultaneously, this implies $ad - bc = 0$ and the proof of necessity is done.

The sufficiency is easily verified. ∎

Proof of Theorem 4: From the proof of Theorem 3 we learn that $\mathbf{p}_M^T(x)\mathbf{R} = [r_1(x)\ r_2(x)\ \cdots\ r_M(x)]$ where $r_k(x) = (a+cx)^{M-k}(b+dx)^{k-1}$, $k = 1, 2, \ldots, M$. Suppose $\mathbf{v}^T$ has a Vandermonde ratio α. When $\alpha \in \mathbb{C}$, $\mathbf{v}^T$ can be expressed as

$$\mathbf{v}^T = g[1 \quad \alpha \quad \alpha^2 \quad \cdots \quad \alpha^{M-1}]$$

for some $g \in \mathbb{C}$. The output $\mathbf{w}^T = \mathbf{v}^T\mathbf{R}$ is thus

$$\mathbf{w}^T = g\,[r_1(\alpha) \quad r_2(\alpha) \quad \cdots \quad r_M(\alpha)].$$

When $a + c\alpha \neq 0$, it is readily verified that the Vandermonde ratio of $\mathbf{w}^T$ is $\beta = r_{k+1}(\alpha)/r_k(\alpha)$ for all k, $1 \leq k \leq M$. This is

$$\beta = \frac{r_{k+1}(\alpha)}{r_k(\alpha)} = \frac{b+d\alpha}{a+c\alpha}.$$

If $a + c\alpha = 0$, then $\mathbf{w}^T = g[0\ \cdots\ 0\ 1]$. So $\beta = \infty$. Finally, when $\alpha = \infty$, $\mathbf{v}^T = g[0\ \cdots\ 0\ 1]$ for some g. So

$$\mathbf{w}^T = g[c^{M-1} \quad c^{M-2}d \quad \cdots \quad d^{M-1}]$$

and $\beta = d/c$ when $c \neq 0$ and $\beta = \infty$ when $c = 0$. In summary,

$$\begin{aligned}\beta &= \begin{cases} \frac{b+d\alpha}{a+c\alpha}, & \text{if } a + c\alpha \neq 0 \\ \infty, & \text{if } a + c\alpha = 0 \text{ and } b + d\alpha \neq 0 \\ \frac{d}{c}, & \text{if } a + c\alpha = 0 \text{ and } b + d\alpha = 0 \end{cases} \\ &= \lim_{x \to \alpha} \frac{b+dx}{a+cx}.\end{aligned}$$

∎

Proof of Lemma 8: Assume $\mathbf{R}$ is not VFP. Then there exists $\mathbf{v}^T \in \mathcal{V}_M$ such that $\mathbf{w}^T = \mathbf{v}^T\mathbf{R} \notin \mathcal{V}_M$. Construct a vector sequence $\mathbf{s}(n)$, $n \geq 0$ as follows. Let $\mathbf{s}(0), \mathbf{s}(1), \ldots, \mathbf{s}(M-2)$ be selected as $(M-1)$ linearly independent column vectors that are orthogonal to $\mathbf{w}^T \notin \mathcal{V}_M$. Let $\mathbf{s}(n) = \mathbf{0}$ for all $n \geq M-1$. Since $\mathbf{w}^T \notin \mathcal{V}_M$ is the only annihilator of $\mathbf{s}(n)$, there does not exist a $2 \times M$ nonzero Hankel matrix $\mathbf{H}$ such that $\mathbf{H}\mathbf{s}(n) = \mathbf{0}$. So $\mathbf{s}(n)$ is (1/2)-rich and hence is $(1/Q)$-rich for any $Q \geq 2$. Now consider $\mathbf{u}(n) = \mathbf{R}\mathbf{s}(n)$. We have $\mathbf{v}^T\mathbf{u}(n) = \mathbf{v}^T\mathbf{R}\mathbf{s}(n) = \mathbf{w}^T\mathbf{s}(n) = \mathbf{0}$. By Lemma 2, $\mathbf{u}(n)$ is not $1/Q$-rich for any Q. So $\mathbf{R}$ is not $(1/Q)$-richness preserving for any $Q \geq 2$. ∎

The proof of Theorem 5 requires the following lemma.

Lemma 11: Let $\mathbf{H}$ be an $m \times n$ Hankel matrix whose entry values come from an $(m+n-1) \times 1$ vector $\mathbf{h}$. That is, $[\mathbf{H}]_{ij} = [\mathbf{h}]_{i+j-1} = h_{i+j-1}$. Let $\mathbf{u}$ and $\mathbf{v}$ be $m \times 1$ and $n \times 1$ column vectors, respectively, and $u(x) = \mathbf{p}_m^T(x)$ and $v(x) = \mathbf{p}_n^T(x)$ are the polynomials representing two vectors. Let $w(x) = u(x)v(x)$ and $\mathbf{w}$ be an $(m+n-1) \times 1$ vector whose polynomial representation is $w(x)$ (i.e., $w(x) = \mathbf{p}_{m+n-1}^T(x)\mathbf{w}$). Then

$$\mathbf{u}^T\mathbf{H}\mathbf{v} = \mathbf{w}^T\mathbf{h}.$$

Proof: The lemma is immediately verified by observing that the coefficient associated with h_k in the sum $\mathbf{u}^T\mathbf{H}\mathbf{v}$ is $\sum_{l=1}^{m} u_l v_{k-l+1}$. (Assuming $v_l = 0$ when $l \leq 0$ or $l > n$.) ∎

Proof of Theorem 5: Denote the kth column of $\mathbf{R}_m$ as $\mathbf{r}_{m,k}$ and the lth column of $\mathbf{R}_n$ as $\mathbf{r}_{n,l}$. Let $r_{mk}(x) = \mathbf{p}_m^T(x)\mathbf{r}_{m,k}$ and $r_{nl}(x) = \mathbf{p}_n^T(x)\mathbf{r}_{n,l}$. From construction of VFP matrices we know $r_{mk}(x) = (a+cx)^{m-k}(b+dx)^{k-1}$ and $r_{nl}(x) = (a+cx)^{n-l}(b+dx)^{l-1}$. The klth entry of $\mathbf{H}'$, $[\mathbf{H}']_{kl}$, can be expressed as $\mathbf{r}_{m,k}^T\mathbf{H}\mathbf{r}_{n,l}$. Using Lemma 11, we have

$$[\mathbf{H}']_{kl} = \mathbf{w}_{k,l}^T\mathbf{h}, \tag{14}$$

where the polynomial representation of the $(m+n-1) \times 1$ vector $\mathbf{w}_{k,l}$ is $w_{kl}(x) = r_{mk}(x)r_{nl}(x) = (a+cx)^{m+n-k-l}(b+dx)^{k+l-2}$. The polynomial $w_{kl}(x)$ stays unchanged when $k+l$ is fixed. So from (14), the value of $[\mathbf{H}']_{kl}$ is a function of $(k+l)$ and hence $\mathbf{H}'$ is also a Hankel matrix. $\mathbf{H}'$ being nonzero is readily verified by observing that both $\mathbf{R}_n$ and $\mathbf{R}_m$ are invertible. ∎

Proof of Lemma 9: In view of proof of Lemma 8, this lemma is self-evident. ∎

Proof of Lemma 10: If $\mathbf{s}(n)$ has rank r, then $\text{rank}(\mathbf{s}_Q(n)) \leq rQ$. Since $\mathbf{s}(n)$ is Q-rich, then $\mathbf{s}_Q(n)$ is rich and hence $\text{rank}(\mathbf{s}_Q(n)) = M + Q - 1 \leq rQ$. So $r \geq (M + Q - 1/Q)$ and hence the proof is complete. ∎

ACKNOWLEDGMENT

The authors would like to appreciate anonymous reviewers who provided useful comments that improved the quality of this paper.

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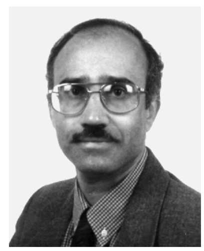
Borching Su (S'00) was born in Tainan, Taiwan, on October 8, 1978. He received the B.S. and M.S. degrees in electrical engineering and communication engineerin from the National Taiwan University (NTU), Taipei, Taiwan, R.O.C., in 1999 and 2001, respectively. He is currently pursuing the Ph.D. degree in the field of digital signal processing at the California Institute of Technology (Caltech), Pasadena.

His current research interests include multirate systems and their applications on digital communications.

Mr. Su was awarded the Moore Fellowship from Caltech in 2003.



P. P. Vaidyanathan (S'80–M'83–SM'88–F'91) was born in Calcutta, India, on October 16, 1954. He received the B.Sc. (Hons.) degree in physics and the B.Tech. and M.Tech. degrees in radiophysics and electronics from the University of Calcutta, Kolkata, India, in 1974, 1977, and 1979, respectively, and the Ph.D degree in electrical and computer engineering from the University of California, Santa Barbara, in 1982.

He was a Postdoctoral Fellow at the University of California, Santa Barbara from September 1982 to March 1983. In March 1983 he joined the Electrical Engineering Department, California Institute of Technology (Caltech), Pasadena, as an Assistant Professor, and since 1993 has been Professor of electrical engineering there. His main research interests are in digital signal processing, multirate systems, wavelet transforms, and signal processing for digital communications. He has authored a number of papers in IEEE journals, and is the author of the book *Multirate Systems and Filter Banks* (Prentice-Hall, 1993). He has written several chapters for various signal processing handbooks. He is a Consulting Editor for the journal *Applied and Computational Harmonic Analysis.*

Dr. Vaidyanathan was a recepient of the Award for Excellence in Teaching at Caltech for the years 1983–1984, 1992–1993, and 1993–1994. He also received the National Science Foundation's Presidential Young Investigator award in 1986. In 1989, he received the IEEE ASSP Senior Award for his paper on multirate perfect-reconstruction filter banks. In 1990, he was the recepient of the S. K. Mitra Memorial Award from the Institute of Electronics and Telecommuncations Engineers, India, for his joint paper in the IETE journal. He was also the coauthor of a paper on linear-phase perfect reconstruction filter banks in the IEEE TRANSACTIONS ON SIGNAL PROCESSING, for which the first author (Truong Nguyen) received the Young Outstanding Author Award in 1993. He received the 1995 F. E. Terman Award of the American Society for Engineering Education, sponsored by Hewlett Packard Co., for his contributions to engineering education, especially the book *Multirate Systems and Filter Banks*. He has given several plenary talks including at the Sampta'01, Eusipco'98, SPCOM'95, and Asilomar'88 conferences on signal processing. He has been chosen a Distinguished Lecturer for the IEEE Signal Processing Society for the year 1996–1997. In 1999, he was chosen to receive the IEEE CAS Society's Golden Jubilee Meda and is a recipient of the IEEE Signal Processing Society's Technical Achievement Award for the year 2002. He served as Vice-Chairman of the Technical Program Committee for the 1983 IEEE International Symposium on Circuits and Systems, and as the Technical Program Chairman for the 1992 IEEE International symposium on Circuits and Systems. He was an Associate editor for the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS for the period 1985–1987 and is currently an Associate Editor for the journal IEEE SIGNAL PROCESSING LETTERS He has been a Guest Editor in 1998 for special issues of the IEEE TRANSACTIONS ON SIGNAL PROCESSING and the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS II, on the topics of filter banks, wavelets, and subband coders.