Properties of the MIMO Radar Ambiguity Function

Chun-Yang Chen and P. P. Vaidyanathan
Dept. of Electrical Engineering, MC 136-93
California Institute of Technology, Pasadena, CA 91125, USA
E-mail: cyc@caltech.edu, ppvnath@systems.caltech.edu

***Abstract*—MIMO (multiple-input multiple-output) radar is an emerging technology which has drawn considerable attention. Unlike the traditional SIMO (single-input multiple-output) radar, which transmits scaled versions of a single waveform in the antenna elements, the MIMO radar transmits independent waveforms in each of the antenna elements. It has been shown that MIMO radar systems have many advantages such as high spatial resolution, improved parameter identifiability, and enhanced flexibility for transmit beampattern design. In the traditional SIMO radar, the range and Doppler resolutions can be characterized by the radar ambiguity function. It is a major tool for studying and analyzing radar signals. Recently, the ambiguity function has been extended to the MIMO radar case. In this paper, some mathematical properties of the MIMO radar ambiguity function are derived. These properties provide insights into the MIMO radar waveform design.** [1]

***Index Terms*— MIMO Radar, Ambiguity Function, Waveform design, Linear Frequency Modulation (LFM).**

I. Introduction

In traditional SIMO (single-input multiple-output) radar systems the antenna elements only emit scaled versions of a single waveform. The MIMO (multiple-input multiple-output) radar is a system which allows independent waveforms from the antenna elements. It has been shown that this kind of radar system has many advantages such as high spatial resolution [2], excellent interference rejection capability [3], improved parameter identifiability [4], and enhanced flexibility for transmit beampattern design [5]. Most of the MIMO radar work so far has ignored the specific details of the waveforms at the antenna elements. However, the choice of the waveforms affects the range, Doppler and angular resolutions of the radar system. In the traditional SIMO radar, the resolution performance of the radar system is characterized by the radar ambiguity function. It is a major tool for studying and analyzing radar signals [6]. Recently, the ambiguity function has been extended to the MIMO radar case [1]. It turns out that the radar waveforms affect not only the range and Doppler resolution but also the angular resolution.

It is well-known that the radar ambiguity function satisfies some properties such as constant energy and symmetry with respect to the origin [6]. These properties are very handy tools for designing and analyzing the radar waveforms. In this paper, we derive the corresponding properties for the case of MIMO radar ambiguity functions. The rest of this paper is organized as follows. Section II reviews the definition of the MIMO radar ambiguity function. Section III derives the properties of the MIMO radar ambiguity function, and Section IV concludes the paper.

[1]Work supported in parts by the ONR grant N00014-06-1-0011, and the California Institute of Technology.

II. Review of MIMO Radar Ambiguity Function

In a SIMO radar system, the radar ambiguity function is defined as [6]

$$|\chi(\tau,\nu)| \triangleq \left| \int_{-\infty}^{\infty} u(t)u^*(t+\tau)e^{j2\pi\nu t}dt \right|, \quad (1)$$

where $u(t)$ is the radar waveform. This two-dimensional function indicates the matched filter output in the receiver when a delay mismatch τ and a Doppler mismatch ν occur. The value $|\chi(0,0)|$ represents the matched filter output without any mismatch. Therefore, the sharper the function $|\chi(\tau,\nu)|$ around $(0,0)$, the better the Doppler and range resolution. Fig. 1 shows two examples of the ambiguity function. These two ambiguity functions show different Doppler and range trade-offs. One can see that the LFM pulse has a better range resolution along the cut where Doppler frequency is zero.

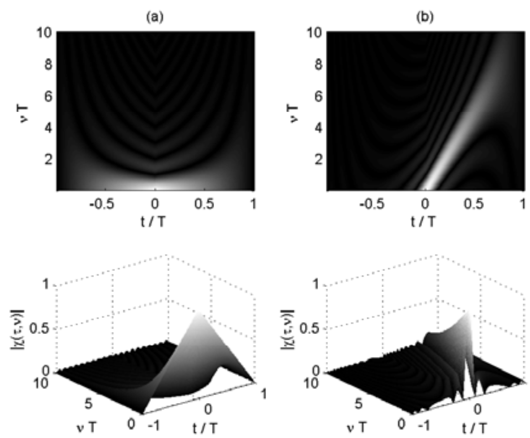

Fig. 1. Examples of ambiguity functions: (a) Rectangular pulse, and (b) Linear frequency modulation (LFM) pulse with time-bandwidth product 10, where T is the pulse duration .

The idea of radar ambiguity functions has been extended to the MIMO radar by San Antonio et al. [1]. In this section, we will briefly review the definition of MIMO radar ambiguity functions. We will focus only on the ULA (uniform linear array) case as shown in Fig. 2. We assume the transmitter and the receiver are parallel ULAs in the same location. The spacing between the transmitting elements is d_T and the spacing between the receiving elements is d_R. The function $u_i(t)$ indicates the radar waveform emitted by the ith transmitter.

Consider a target at (τ,ν,f_s) where τ is the delay corresponding to the target range, ν is the Doppler frequency of the target, and f_s is the normalized spatial frequency of the

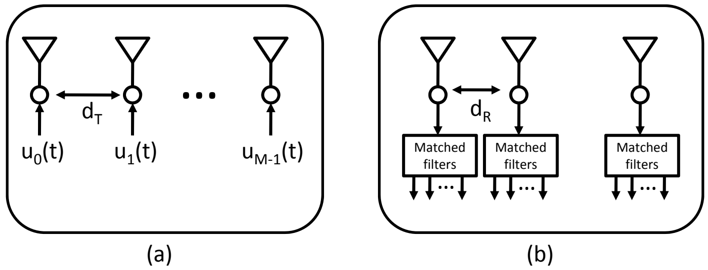

Fig. 2. MIMO radar scheme: (a) Transmitter, and (b) Receiver.

target. The demodulated target response in the nth antenna is proportional to

$$y_n^{\tau,\nu,f_s}(t) \approx \sum_{m=0}^{M-1} u_m(t-\tau)e^{j2\pi\nu t}e^{j2\pi f_s(\gamma m+n)},$$

for $n = 0, 1, \cdots, N-1$, where N is the number of receiving antennas, $u_m(t)$ is the radar waveform emitted by the mth antenna, $\gamma \triangleq d_T/d_R$ and M is the number of transmitting antennas. If the receiver tries to capture this target signal with a matched filter with the assumed parameters (τ', ν', f_s') then the matched filter output becomes

$$\begin{aligned}
&\sum_{n=0}^{N-1}\int_{-\infty}^{\infty} y_n^{\tau,\nu,f_s}(t)\cdot(y_n^{\tau',\nu',f_s'})^*(t)dt \\
= &\left(\sum_{n=0}^{N-1} e^{j2\pi(f_s-f_s')n}\right)\cdot \\
&\left(\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\int_{-\infty}^{\infty} u_m(t-\tau)u_{m'}^*(t-\tau')\right. \\
&\left. e^{j2\pi(\nu-\nu')t}dt\cdot e^{j2\pi(f_s m-f_s' m')\gamma}\right)
\end{aligned}$$

The first part in the right hand side of the equation represents the spatial processing in the receiver, and it is not affected by the waveforms $\{u_m(t)\}$. The second part in the right hand side of the equation indicates how the waveforms $\{u_m(t)\}$ affect the spatial, Doppler and range resolutions of the radar system. Therefore, we define the **MIMO radar ambiguity function** as

$$\chi(\tau,\nu,f_s,f_s') \triangleq \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\chi_{m,m'}(\tau,\nu)e^{j2\pi(f_s m-f_s' m')\gamma}, \quad (2)$$

where

$$\chi_{m,m'}(\tau,\nu) \triangleq \int_{-\infty}^{\infty} u_m(t)u_{m'}^*(t+\tau)e^{j2\pi\nu t}dt. \quad (3)$$

Note that the MIMO radar ambiguity function can not be expressed as a function of the difference of the spatial frequencies, namely $f_s - f_s'$. Therefore, we need both the target spatial frequency f_s and the assumed spatial frequency f_s' to represent the spatial mismatch. We call the function $\chi_{m,m'}(\tau,\nu)$ the **cross ambiguity function** because it is similar to the SIMO ambiguity function defined in (1) except it involves two waveforms $u_m(t)$ and $u_{m'}(t)$. Fixing τ and ν in (2), one can view the ambiguity function as a scaled two-dimensional Fourier transform of the cross ambiguity function $\chi_{m,m'}(\tau,\nu)$ on the parameters m and m'. The value $|\chi(0,0,f_s,f_s)|$ represents the matched filter output without mismatch. Therefore, the sharper the function $|\chi(\tau,\nu,f_s,f_s')|$ around the line $\{(0,0,f_s,f_s)\}$, the better the radar system resolution.

III. Properties of The MIMO Radar Ambiguity Function

We now derive some new properties of the MIMO radar ambiguity function defined in (2). The properties are similar to some of the properties of the SIMO ambiguity functions (e.g., see [6]). We normalize the energy of the transmitted waveform to unity. That is,

$$\int_{-\infty}^{\infty} |u_m(t)|^2 dt = 1, \forall m \quad (4)$$

The following property characterizes the ambiguity function when there exists no mismatch.

Property 1. If $\int_{-\infty}^{\infty} u_m(t)u_{m'}^*(t)dt = \delta_{m,m'}$, then

$$\chi(0,0,f_s,f_s) = M, \forall f_s.$$

Proof: We have

$$\chi_{m,m'}(0,0) = \int_{-\infty}^{\infty} u_m(t)u_{m'}^*(t)dt = \delta_{m,m'}.$$

Substituting the above equation into (2), we obtain

$$\begin{aligned}
\chi(0,0,f_s,f_s) &= \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\delta_{m,m'}e^{j2\pi\gamma(f_s m-f_s m')} \\
&= \sum_{m=0}^{M-1} e^{j0} = M. \qquad \blacksquare
\end{aligned}$$

This property says that if the waveforms are orthogonal, the ambiguity function is a constant along the line $\{(0,0,f_s,f_s)\}$ which is independent of the waveforms $\{u_m(t)\}$. This means the matched filter output is always a constant independent of the waveforms, when there exists no mismatch.

The following property characterizes the integration of the MIMO radar ambiguity function along the line $\{0,0,f_s,f_s\}$ even when the waveforms are not orthogonal.

Property 2.

$$\chi(0,0,f_s,f_s) \geq 0,$$

and if γ is an integer, then

$$\int_0^1 \chi(0,0,f_s,f_s)df_s = M.$$

Proof: By using the definitions in (2) and (3), we have

$$\chi(0,0,f_s,f_s) = \int_{-\infty}^{\infty} \left| \sum_{m=0}^{M-1} u_m(t) e^{j2\pi f_s m\gamma} \right|^2 dt \geq 0$$

By using the definitions in (2) and (3) and changing variable, we obtain

$$\begin{aligned} &\int_0^1 \chi(0,0,f_s,f_s)df_s \\ = &\int_0^1 \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \chi_{m,m'}(0,0,)e^{j2\pi f_s \gamma(m-m')} df_s \\ = &\sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} \chi_{m,m'}(0,0,)\delta_{m,m'} = M \end{aligned}$$

■

This property says that when γ is an integer, the integration of the MIMO radar ambiguity function along the line $\{0,0,f_s,f_s\}$ is a constant, no matter how waveforms are chosen. The following property characterizes the energy of the cross ambiguity function.

Property 3.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi_{m,m'}(\tau,\nu)|^2 d\tau d\nu = 1.$$

Proof: We have

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi_{m,m'}(\tau,\nu)|^2 d\tau d\nu \\ = &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} u_m(t) u_{m'}^*(t+\tau) e^{j2\pi\nu t} dt \right|^2 d\nu d\tau \\ = &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| u_m(t) u_{m'}^*(t+\tau) \right|^2 dt d\tau, \end{aligned}$$

where we have used Parseval's theorem [7] to obtain the last equality. By changing variables, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| u_m(t) u_{m'}^*(t+\tau) \right|^2 dt d\tau &= \\ \int_{-\infty}^{\infty} |u_m(t)|^2 dt \int_{-\infty}^{\infty} |u_{m'}(t)|^2 dt &= 1 \end{aligned}$$

■

This property states that the energy of the cross ambiguity function is a constant, independent of the waveforms $u_m(t)$ and $u_{m'}(t)$. In the special case of $m = m'$, this property implies that the SIMO radar ambiguity function defined in (1) has constant energy [6]. The following property characterizes the energy of the MIMO radar ambiguity function.

Property 4. If γ is an integer, then

$$\int_0^1 \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(\tau,\nu,f_s f_s')|^2 d\tau d\nu df_s df_s' = M^2.$$

Proof: By using the definition of MIMO radar ambiguity function in (2) and performing appropriate change of variables, we have

$$\begin{aligned} &\int_0^1 \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(\tau,\nu,f_s f_s')|^2 d\tau d\nu df_s df_s' \\ = &\frac{1}{\gamma^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\gamma} \int_0^{\gamma} \\ &\left| \sum_{m=0}^{M-1} \sum_{m=0}^{M-1} \chi_{m,m'}(\tau,\nu) e^{j2\pi(f_s m - f_s' m')} \right|^2 df_s df_s' d\tau d\nu \end{aligned} \quad (5)$$

Using Parseval's theorem and applying Property 3, the above integral reduces to

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} |\chi_{m,m'}(\tau,\nu)|^2 d\tau d\nu = \sum_{m'=0}^{M-1} \sum_{m'=0}^{M-1} 1 = M^2$$

■

This property states that when γ is an integer, the energy of the MIMO radar ambiguity function is a constant which is independent of the waveforms $\{u_m(t)\}$. For example, whether we choose $\gamma = 1$ or $\gamma = N$, the energy of the MIMO radar ambiguity function is the same. Recall that Property 2 states that the integration of MIMO radar ambiguity function along the line $\{(0,0,f_s,f_s\}$ is also a constant. This implies that in order to make the ambiguity function sharp around $\{0,0,f_s,f_s\}$, we have to spread the energy of the ambiguity function evenly on the available time and bandwidth.

For the case that γ is not an integer, we can not directly apply Parseval's theorem. In this case, the energy of the ambiguity function actually depends on the waveforms $\{u_m(t)\}$. However, the following property characterizes the range of the energy of the MIMO radar ambiguity function.

Property 5.

$$\begin{aligned} \frac{\lfloor\gamma\rfloor^2}{\gamma^2} M^2 \leq \int_0^1 \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(\tau,\nu,f_s f_s')|^2 d\tau d\nu df_s df_s' \\ \leq \frac{\lceil\gamma\rceil^2}{\gamma^2} M^2 \end{aligned}$$

where $\lfloor\gamma\rfloor$ is the largest integer $\leq \gamma$, and $\lceil\gamma\rceil$ is the smallest integer $\geq \gamma$.

Proof: Using (5), we have

$$\begin{aligned} &\int_0^1 \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(\tau,\nu,f_s f_s')|^2 d\tau d\nu df_s df_s' \\ \leq &\frac{1}{\gamma^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\lceil\gamma\rceil} \int_0^{\lceil\gamma\rceil} \\ &\left| \sum_{m=0}^{M-1} \sum_{m=0}^{M-1} \chi_{m,m'}(\tau,\nu) e^{j2\pi(f_s m - f_s' m')} \right|^2 df_s df_s' d\tau d\nu \end{aligned}$$

Using Parseval's theorem and applying Property 3, the above value equals

$$\frac{\lceil\gamma\rceil^2}{\gamma^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=0}^{M-1} \sum_{m'=0}^{M-1} |\chi_{m,m'}(\tau,\nu)|^2 d\tau d\nu = \frac{\lceil\gamma\rceil^2}{\gamma^2} M^2$$

The lower bound can be obtained similarly. ■

For the case that γ is not integer, the energy of the MIMO radar ambiguity function can actually be affected by the waveforms $\{u_m(t)\}$. However, the above property implies that the amount of the energy which can be affected by the waveforms is small. Using similar lines of argument, we can show that when γ is not an integer, Property 2 can be replaced with

$$M\frac{\lfloor\gamma\rfloor}{\gamma} \leq \int_0^1 \chi(0,0,f_s,f_s)df_s \leq M\frac{\lceil\gamma\rceil}{\gamma}.$$

The following property characterizes the symmetry of the cross ambiguity function.

Property 6.

$$\chi_{m,m'}(-\tau,-\nu) = \chi^*_{m',m}(\tau,\nu)e^{-j2\pi\nu\tau}$$

Proof: By the definition of the cross ambiguity function (3) and changing variables, we have

$$\begin{aligned}\chi_{m,m'}(-\tau,-\nu) &= \int_{-\infty}^{\infty} u_m(t)u^*_{m'}(t-\tau)e^{-j2\pi\nu t}dt \\ &= \int_{-\infty}^{\infty} u_m(t+\tau)u^*_{m'}(t)e^{-j2\pi\nu(t+\tau)}dt \\ &= \chi^*_{m',m}(\tau,\nu)e^{-j2\pi\nu\tau} \quad \blacksquare\end{aligned}$$

Using the above property, we can obtain the following property of the MIMO radar ambiguity function.

Property 7.

$$\chi(-\tau,-\nu,f_s,f'_s) = \chi^*(\tau,\nu,f'_s,f_s)e^{-j2\pi\nu\tau}$$

Proof: Using the definition of the MIMO radar ambiguity function (2) and Property 6, we have

$$\begin{aligned}&\chi(-\tau,\nu,f_s,f'_s) \\ =& \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\chi_{m,m'}(-\tau,-\nu)e^{j2\pi\gamma(f_s m - f'_s m')} \\ =& \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\chi^*_{m',m}(\tau,\nu)e^{-j2\pi\nu\tau}e^{j2\pi\gamma(f_s m - f'_s m')} \\ =& \left(\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\chi_{m',m}(\tau,\nu)e^{j2\pi\gamma(f'_s m' - f_s m)}\right)^* e^{-j2\pi\nu\tau} \\ =& \chi^*(\tau,\nu,f'_s,f_s)e^{-j2\pi\nu\tau} \quad \blacksquare\end{aligned}$$

This property implies that when we design the waveform, we only need to focus on the region $\{(\tau,\nu,f_s,f'_s)|\nu \geq 0\}$ or the region $\{(\tau,\nu,f_s,f'_s)|f_s \geq f'_s\}$ of the MIMO radar ambiguity function. For example, given two spatial frequency f_s and f'_s it is sufficient to study only $\chi(\tau,\nu,f_s,f'_s)$ because the function $\chi(\tau,\nu,f'_s,f_s)$ can be deduced from the symmetry property. The following property characterizes the cross ambiguity function of the linear frequency modulation (LFM) signal.

Property 8. If $\chi_{m,m'}(\tau,\nu) = \int_{-\infty}^{\infty} u_m(t)u^*_{m'}(t+\tau)e^{j2\pi\nu t}dt$ then

$$\begin{aligned}\chi^{LFM}_{m,m'}(\tau,\nu) &\triangleq \int_{-\infty}^{\infty}\left(u_m(t)e^{j\pi k\nu t^2}\right)\cdot \\ &\quad \left(u_{m'}(t+\tau)e^{j\pi k\nu(t+\tau)^2}\right)^* e^{j2\pi\nu t}dt \\ &= \chi_{m,m'}(\tau,\nu-k\tau)e^{-j\pi k\tau^2}\end{aligned}$$

Proof: From direct calculation, we have

$$\begin{aligned}\chi^{LFM}_{m,m'}(\tau,\nu) &= \int_{-\infty}^{\infty} u_m(t)u^*_{m'}(t+\tau)\cdot \\ &\quad e^{j\pi k(-2t\tau+\tau^2)}e^{-j2\pi\nu t}dt \\ &= \chi_{m,m'}(\tau,\nu-k\tau)e^{-j\pi k\tau^2} \quad \blacksquare\end{aligned}$$

This property says that linear frequency modulation shears off the cross ambiguity function. We use this property to obtain the following result for the MIMO radar ambiguity function.

Property 9.
If $\chi(\tau,\nu,f_s,f'_s) = \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\chi_{m,m'}(\tau,\nu)e^{j2\pi\gamma(f_s m - f'_s m')}$ then

$$\begin{aligned}\chi^{LFM}(\tau,\nu,f_s,f'_s) &\triangleq \sum_{m=0}^{M-1}\sum_{m'=0}^{M-1}\chi^{LFM}_{m,m'}(\tau,\nu)e^{j2\pi\gamma(f_s m - f'_s m')} \\ &= \chi(\tau,\nu-k\tau,f_s,f'_s)e^{jk\nu\tau^2}\end{aligned}$$

We omit the proof because this property can be easily proven by just applying Property 8. This property states that adding LFM modulations shears off the MIMO radar ambiguity function. This shearing can improve the range resolution because it compresses the ambiguity function along the direction $(\tau,0,f_s,f_s)$ [6].

IV. Conclusions

In this paper, we have derived several properties of the MIMO radar ambiguity function and the cross ambiguity function. These results are derived for the ULA case. To summarize, Property 1 and 2 characterize the MIMO radar ambiguity function along the line $\{(0,0,f_s,f_s)\}$. Properties 3, 4, and 5 characterize the energy of the cross ambiguity function and the MIMO radar ambiguity function. These properties imply that we can only spread the energy of the MIMO radar ambiguity function evenly on the available time and bandwidth because the energy is confined. Properties 6 and 7 show the symmetry of the cross ambiguity function and the MIMO radar ambiguity function. These properties imply that when we design the waveform, we only need to focus on the region $\{(\tau,\nu,f_s,f'_s)|\nu \geq 0\}$ of the MIMO radar ambiguity function. Property 8 and 9 show the shear-off effect of the LFM waveform. This shearing improves the range resolution. We believe these properties will be helpful for designing and analyzing the MIMO radar waveforms.

References

[1] G. San Antonio, D. R. Fuhrmann, and F. C. Robey, "MIMO Radar Ambiguity Functions," *IEEE Journal of Selected Topics in Signal Processing,* vol. 1, pp. 167-177, Jun. 2007.

[2] D. W. Bliss and K. W. Forsythe, "Multiple-input multiple-output (MIMO) radar and imaging: degrees of freedom and resolution," *Proc. 37th IEEE Asilomar Conf. on Signals, Systems, and Computers,* vol. 1, pp. 54-59, Nov. 2003.

[3] Chun-Yang Chen and P. P. Vaidyanathan, "MIMO Radar Space-Time Adaptive Processing Using Prolate Spheroidal Wave Functions," *IEEE Trans. on Signal Processing*, to appear.

[4] J. Li, P. Stoica, L. Xu, and W. Roberts, "On Parameter Identifiability of MIMO Radar," *IEEE Signal Processing Letters*, to appear.

[5] P. Stoica, J. Li, and Y. Xie, "On Probing Signal Design For MIMO Radar," *IEEE Trans. on Signal Processing* Volume 55, Issue 8, Aug. 2007.

[6] N. Levanon and E. Mozeson, *Radar Signals,* Wiley-IEEE Press 2004.

[7] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing,* Prentice Hall, 1998.