Contour integration method for inverse z-transformation

The z-transform is defined by the summation

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}, \tag{1}$$

and converges in an annular region $R_1 < |z| < R_2$. Expressing the complex number z as $z = Re^{j\omega}$ where R is the radius and ω the phase of z, we have

$$X(z) = \sum_{n=-\infty}^{\infty} \left[R^{-n} x(n)\right] e^{-j\omega n} \tag{2}$$

which shows that $X(z)$ at $z = Re^{j\omega}$ is nothing but the Fourier transform of the modified sequence

$$R^{-n} x(n).$$

If $X(z)$ is known everywhere on a circle of radius R (i.e., $X(z)$ known for all ω for fixed R) in the ROC, then we can use the inverse *Fourier* transform relation to write

$$R^{-n} x(n) = \int_{-\infty}^{\infty} X(z) e^{j\omega n} \frac{d\omega}{2\pi} \tag{3}$$

From the relation $z = Re^{j\omega}$ we have

$$dz = jRe^{j\omega} d\omega = jz d\omega \tag{4}$$

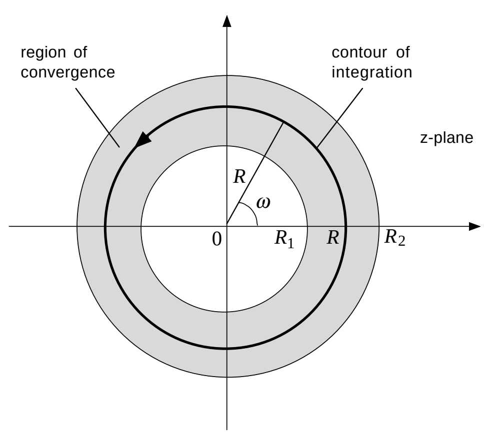
for fixed R, so that (3) can be rewritten as

$$R^{-n} x(n) = \int_{-\infty}^{\infty} X(z) e^{j\omega n} \frac{dz}{2\pi j z} \tag{5}$$

Taking R^{-n} to the other side and writing $R^n e^{j\omega n} = z^n$ we therefore get

$$x(n) = \frac{1}{2\pi j} \oint_{|z|=R} X(z) z^{n-1} dz \tag{6}$$

Notice that the integration with respect to ω has been rewritten as a **contour** integral in the z-plane. The contour is any circle of radius R in the region of convergence. Going around this contour (counterclockwise) once is the same as increasing the frequency ω (the angle indicated in the figure below) from 0 to 2π.



over⇒

Residues

The contour integral can be evaluated using standard techniques from complex variable theory. There is a theorem due to Cauchy which says that if $f(z)$ is single-valued and analytic on and inside a closed contour $\mathcal{C}$ except for a finite number of poles $p_1, p_2, \ldots, p_N$ inside $\mathcal{C}$, then

$$\frac{1}{2\pi j} \oint_{\mathcal{C}} f(z)dz = \sum_{k=1}^{N} R_k \tag{7}$$

where R_k is the so-called **residue** of $f(z)$ at the pole p_k. When p_k is a simple pole (as against a multiple pole such as a double pole) then the residue is calculated as follows:

$$R_k = \lim_{z \to p_k} (z - p_k) f(z) \tag{8}$$

If p_k is a pole of order M, then the residue is calculated using a more complicated formula:

$$R_k = \lim_{z \to p_k} \frac{1}{(M-1)!} \frac{d^{M-1}}{dz^{M-1}} \left[(z - p_k)^M f(z) \right] \tag{9}$$

So, given a rational function $X(z)$ and a specified region of convergence $R_1 < |z| < R_2$, we can compute the inverse z-transform $x(n)$ as follows:

1. Make a list of the poles of $z^{n-1}X(z)$ in the region $0 \leq |z| < R_1$.
2. Compute the residues of these poles and add them up. The result is precisely $x(n)$.

This method is called the *contour integration method* for computing the inverse z-transform. It can be somewhat cumbersome because of the poles created at $z = 0$ by the factor z^{n-1} in Eq. (6) (when $n < -1$). These are multiple poles and the multiplicity $n - 1$ depends on the time index n. Calculating the residues for these can be laborious. But there are examples where the contour integration method is the most convenient. For example problem 3.39 in AVO's book is easy to solve using contour integration.