**EE 150 - Applications of Convex Optimization in Signal Processing and Communications**
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# Homework Set #3

Due on **Thursday, April 26** in class.

**1.** (10 points) (Adapted from CO-BV, Exercise 2.9) (*Voronoi sets and polyhedral decomposition:*) Let $\mathbf{x}_0, \ldots, \mathbf{x}_K \in \mathbb{R}^n$ denote a set of $(K+1)$ real vectors. Consider the set of points in $\mathbb{R}^n$ that are closer (in the Euclidean norm sense) to $\mathbf{x}_0$ than the other vectors from above. Specifically, consider the set

$$\mathcal{V} \triangleq \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{x}_0||_2 \leq ||\mathbf{x} - \mathbf{x}_\ell||_2 \ , \ \ell = 1, \ldots, K\} \ .$$

The set $\mathcal{V}$ is called the *Voronoi region* around $\mathbf{x}_0$ with respect to $\mathbf{x}_1, \ldots, \mathbf{x}_K$.

(a) Show that $\mathcal{V}$ is a polyhedron. Specifically, express $\mathcal{V}$ in the form $\mathcal{V} = \{\mathbf{x} : \mathbf{A}\mathbf{x} \preceq \mathbf{b}\}$.

(b) Conversely, given a polyhedron $\mathcal{P}$ with nonempty interior, show how to find a set of points $\mathbf{x}_0, \ldots, \mathbf{x}_K$ so that the polyhedron is the Voronoi region of $\mathbf{x}_0$ with respect to $\mathbf{x}_1, \ldots, \mathbf{x}_K$.

(c) Similar to the Voronoi region $\mathcal{V}$ defined above, we can also consider the following sets:

$$\mathcal{V}_k = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{x}_k||_2 \leq ||\mathbf{x} - \mathbf{x}_\ell||_2 \ , \ \ell \neq k\} \ , \ k = 0, \ldots, K \ .$$

The set $\mathcal{V}_k$ consists of points in $\mathbb{R}^n$ for which the closest point in the set $\{\mathbf{x}_0, \ldots, \mathbf{x}_K\}$ is $\mathbf{x}_k$. (Note that we have $\mathcal{V}_0 = \mathcal{V}$ here.)
The sets $\mathcal{V}_0, \ldots, \mathcal{V}_K$ give a polyhedral decomposition of $\mathbb{R}^n$. More precisely, the sets $\mathcal{V}_k$ are polyhedra and we have

$$\bigcup_{k=0}^{K} \mathcal{V}_k = \mathbb{R}^n \ , \ \operatorname{int}(\mathcal{V}_k) \cap \operatorname{int}(\mathcal{V}_\ell) = \varnothing \text{ for } k \neq \ell \ .$$

In other words, the polyhedra $\mathcal{V}_k$ taken together comprise the whole vector space $\mathbb{R}^n$ and are such that $\mathcal{V}_k$ and $\mathcal{V}_\ell$ intersect at most along a boundary for any $k \neq \ell$.
Suppose now that $\mathcal{P}_1, \ldots, \mathcal{P}_m$ are polyhedra such that

$$\bigcup_{k=1}^{m} \mathcal{P}_k = \mathbb{R}^n \ , \ \operatorname{int}(\mathcal{P}_k) \cap \operatorname{int}(\mathcal{P}_\ell) = \varnothing \text{ for } k \neq \ell \ .$$

Can this polyhedral decomposition of $\mathbb{R}^n$ be described as the Voronoi regions generated by an appropriate set of points? If so, prove this statement and if not, show a counterexample.

**2.** (10 points) (Adapted from CO-BV, Exercise 2.10) (*Solution set of a quadratic inequality:*) Let $\mathcal{C} \subseteq \mathbb{R}^n$ be the solution set of the following quadratic inequality:

$$\mathcal{C} \triangleq \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c \leq 0\right\} \ .$$

Here, $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

(a) Show that $\mathcal{C}$ is convex if $\mathbf{A} \succeq \mathbf{0}$. Also, show via a counterexample that the converse is false.

(b) Show that the intersection of $\mathcal{C}$ with the hyperplane $\mathcal{H}$ defined as

$$\mathcal{H} \triangleq \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{g}^T \mathbf{x} + h = 0 \right\} ,$$

where $\mathbf{g} \in \mathbb{R}^n$ with $\mathbf{g} \neq \mathbf{0}$ and $h \in \mathbb{R}$, is convex if $\left(\mathbf{A} + \lambda \mathbf{g}\mathbf{g}^T\right) \succeq \mathbf{0}$ for some $\lambda \in \mathbb{R}$. Also, show via a counterexample that the converse is false.

**3.** (10 points) (Adapted from CO-BV, Exercise 2.15) (*Some sets of probability distributions:*) Let $X$ be a real-valued discrete random variable with $\Pr\{X = a_k\} = p_k$ for $k = 1, \ldots, n$, where $a_1 < \cdots < a_n$. Also, let $\mathbf{p} \triangleq \left[\begin{array}{ccc} p_1 & \cdots & p_n \end{array}\right]^T$ denote the associated vector of probabilities. Evidently $\mathbf{p} \in \mathbb{R}^n$ and also lies in the standard probability simplex $\mathcal{P} \triangleq \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{1}^T \mathbf{x} = 1 ,\ \mathbf{x} \succeq \mathbf{0} \right\}$. Which of the following conditions are convex in $\mathbf{p}$? More specifically, for which of the following conditions is the set of $\mathbf{p} \in \mathcal{P}$ that satisfy the condition convex?

(a) $\alpha \leq E[f(X)] \leq \beta$, where $E[f(X)]$ is the expected value of $f(X)$ given by

$$E[f(X)] = \sum_{k=1}^{n} p_k f(a_k) .$$

The function $f : \mathbb{R} \to \mathbb{R}$ is given here.

(b) $\Pr\{X > \alpha\} \leq \beta$.

(c) $E\left[|X|^3\right] \leq \alpha E[|X|]$.

(d) $E\left[X^2\right] \leq \alpha$.

(e) $E\left[X^2\right] \geq \alpha$.

(f) $\mathrm{Var}(X) \leq \alpha$, where $\mathrm{Var}(X) \triangleq E\left[(X - E[X])^2\right]$ is the *variance* of $X$.

(g) $\mathrm{Var}(X) \geq \alpha$.

(h) $Q_1(X) \geq \alpha$, where $Q_1(X)$ is the *first quartile* of $X$ defined by

$$Q_1(X) = \inf \left\{ \beta : F_X(\beta) \geq \frac{1}{4} \right\} ,$$

and $F_X(x)$ is the *cumulative distribution function* (cdf) of $X$ given by $F_X(x) \triangleq \Pr\{X \leq x\}$. In other words, the first quartile represents the smallest value of $X$ for which all values less than or equal to it account for at least 25% of the total probability.
*Hint:* The illustration in Figure 1 shows how to calculate the quantile from the cdf. From the example shown in this figure, it is clear that $Q_1(X) = a_2$ in this case.

(i) $Q_1(X) \leq \alpha$.

**4.** (10 points) (Adapted from CO-BV, Exercise 3.18) (*Convexity/concavity of common functions of matrices:*) Suppose $\mathbf{X} \in \mathbb{R}^{n \times n}$. Show the following results.

(a) $f(\mathbf{X}) = \mathrm{tr}\left(\mathbf{X}^{-1}\right)$ is convex on $\mathrm{dom}(f) = \mathbb{S}^n_{++}$.

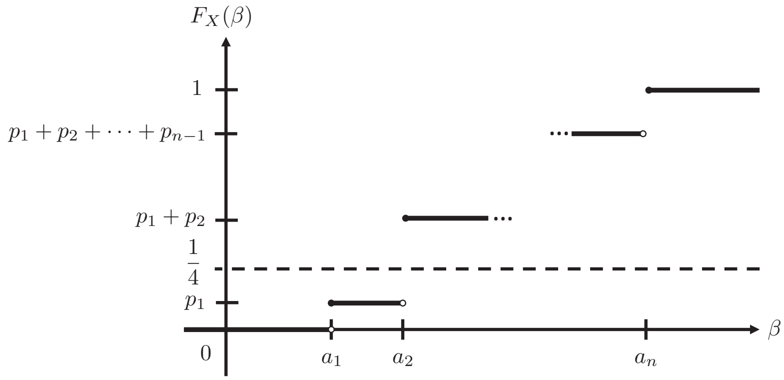


Figure 1: Illustration showing how to compute the quantile $Q_1(X)$ from the cdf $F_X(\beta)$. Here, $Q_1(X) = a_2$.

(b) $f(\mathbf{X}) = (\det(\mathbf{X}))^{\frac{1}{n}}$ is concave on $\mathrm{dom}(f) = \mathbb{S}^n_{++}$.

*Hint:* For both cases, determine the convexity/concavity of the matrix function under consideration by considering the convexity/concavity of the scalar function $g(t) \triangleq f(\mathbf{Z} + t\mathbf{V})$, where $\mathbf{Z} \succ \mathbf{0}$ and $\mathbf{V} \in \mathbb{S}^n$. Then, with the use of an appropriate eigenvalue decomposition, the results will follow.

***5.** (30 points) (Adapted from CO-BV, Exercise 3.25) (*Maximum probability distance between distributions:*) Let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ represent two probability distributions on the set $\{1, \ldots, n\}$ (so that $\mathbf{p}, \mathbf{q} \succeq \mathbf{0}$ and $\mathbf{1}^T\mathbf{p} = \mathbf{1}^T\mathbf{q} = 1$). We define the *maximum probability distance* $d_{\mathrm{mp}}(\mathbf{p}, \mathbf{q})$ between $\mathbf{p}$ and $\mathbf{q}$ as the maximum difference in probability assigned by $\mathbf{p}$ and $\mathbf{q}$ over all possible events. In other words, we have

$$d_{\mathrm{mp}}(\mathbf{p}, \mathbf{q}) \triangleq \max \left\{ |P_{\mathbf{p}}(\mathcal{C}) - P_{\mathbf{q}}(\mathcal{C})| : \mathcal{C} \subseteq \{1, \ldots, n\} \right\} .$$

Here, $P_{\mathbf{p}}(\mathcal{C})$ is the probability of $\mathcal{C}$ under the distribution $\mathbf{p}$, i.e., we have

$$P_{\mathbf{p}}(\mathcal{C}) \triangleq \sum_{k \in \mathcal{C}} p_k .$$

Find a simple expression for $d_{\mathrm{mp}}(\mathbf{p}, \mathbf{q})$, involving $||\mathbf{p} - \mathbf{q}||_1$, and show that $d_{\mathrm{mp}}(\mathbf{p}, \mathbf{q})$ is a convex function on $\mathbb{R}^n \times \mathbb{R}^n$. (Its domain is $\left\{ (\mathbf{p}, \mathbf{q}) : \mathbf{p}, \mathbf{q} \succeq \mathbf{0}, \mathbf{1}^T\mathbf{p} = \mathbf{1}^T\mathbf{q} = 1 \right\}$, but it has a natural extension to all of $\mathbb{R}^n \times \mathbb{R}^n$.)

***Reading assignments:***

**1.** Read through Chapter 2 and begin Chapter 3 of CO-BV.

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**Reminders:**

***Late homework policy for EE 150:*** Late homeworks will not be accepted. There will be no exceptions to this other than institute established emergency reasons, in which case a signed letter is required from an authorized official.

***NCT Problems:*** Remember that problems with an asterisk, such as ***7** are no collaboration type (NCT) problems.

***Texts:*** The abbreviation CO-BV corresponds to the textbook "*Convex Optimization*" by Stephen Boyd and Lieven Vandenberghe. In addition, CO-AE refers to the *Additional Exercises for Convex Optimization*, also by Boyd and Vandenberghe. Finally, CVX corresponds to the `cvx` *Users' Guide* by Michael Grant and Stephen Boyd.

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