EE 150 - Applications of Convex Optimization in Signal Processing and Communications
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Homework Set #5

Due on **Friday, May 11** at 1 PM in 110 Moore.

1. (10 points) (Adapted from CO-BV, Exercise 4.20) (*Power assignment in a wireless communication system:*) We consider n transmitters with powers $p_1, \ldots, p_n \geq 0$, transmitting to n receivers. These powers are the optimization variable in the problem. We let $\mathbf{G} \in \mathbb{R}^{n \times n}$ denote the matrix of *path gains* from the transmitters to the receivers; $G_{i,j} \geq 0$ is the path gain from transmitter j to receiver i. The *signal power* at receiver i is then $S_i = G_{i,i} p_i$, and the *interference power* at receiver i is $I_i = \sum_{k \neq i} G_{i,k} p_k$. The *signal-to-interference-plus-noise ratio*, denoted SINR, at receiver i is given by $S_i / (I_i + N_i)$, where $N_i > 0$ is the (self-) noise power in receiver i. The objective in the problem is to maximize the minimum SINR ratio over all receivers, i.e., to maximize

$$\min_{i=1,\ldots,n} \frac{S_i}{I_i + N_i} .$$

There are a number of constraints on the powers that must be satisfied, in addition to the obvious one $p_i \geq 0$. The first is a maximum allowable power for each transmitter, i.e., $p_i \leq P_i^{\max}$, where $P_i^{\max} > 0$ is given. In addition, the transmitters are partitioned into groups, with each group sharing the same power supply, so there is a total power constraint for each group of transmitter powers. More precisely, we have subsets $\mathcal{K}_1, \ldots, \mathcal{K}_m$ of $\{1, \ldots, n\}$ with $\mathcal{K}_1 \cup \cdots \cup \mathcal{K}_m = \{1, \ldots, n\}$ and $\mathcal{K}_j \cap \mathcal{K}_l = 0$ if $j \neq l$. For each group $\mathcal{K}_l$, the total associated transmitter power cannot exceed $P_l^{\text{gp}} > 0$:

$$\sum_{k \in \mathcal{K}_l} p_k \leq P_l^{\text{gp}} , \ l = 1, \ldots, m .$$

Finally, we have a limit $P_k^{\text{rc}} > 0$ on the total received power at each receiver:

$$\sum_{k=1}^{n} G_{i,k} p_k \leq P_i^{\text{rc}} , \ i = 1, \ldots, n .$$

(This constraint reflects the fact that the receivers will saturate if the total received power is too large.)
Formulate the SINR maximization problem as a generalized linear-fractional program.

2. (10 points) (Adapted from CO-BV, Exercise 4.28) (*Robust quadratic programming:*) In lecture, we discussed robust linear programming as an application of second-order cone programming. For this problem, we consider a similar robust variation of the (convex) *quadratic* program

$$\begin{array}{ll} \text{minimize} & (1/2)\, \mathbf{x}^T \mathbf{P} \mathbf{x} + \mathbf{q}^T \mathbf{x} + r \\ \text{subject to} & \mathbf{A}\mathbf{x} \preceq \mathbf{b} \end{array} .$$

For simplicity, we assume that only the matrix $\mathbf{P}$ is subject to errors, while the other parameters $(\mathbf{q}, r, \mathbf{A}, \mathbf{b})$ are known exactly. The robust quadratic program is defined as

$$\begin{array}{ll} \text{minimize} & \sup_{\mathbf{P} \in \mathcal{E}} \left\{ (1/2)\, \mathbf{x}^T \mathbf{P} \mathbf{x} + \mathbf{q}^T \mathbf{x} + r \right\} \\ \text{subject to} & \mathbf{A}\mathbf{x} \preceq \mathbf{b} \end{array} ,$$

where $\mathcal{E}$ is the set of possible matrices $\mathbf{P}$.
For each of the following sets $\mathcal{E}$, express the robust QP as a convex problem. Be as specific as you can. If the problem can be expressed in a standard form (e.g., QP, QCQP, SOCP, SDP), say so.

(a) A finite set of matrices: $\mathcal{E} = \{\mathbf{P}_1, \ldots, \mathbf{P}_K\}$, where $\mathbf{P}_i \in \mathbb{S}^n_+$, $i = 1, \ldots, K$.

(b) A set specified by a nominal value $\mathbf{P}_0 \in \mathbb{S}^n_+$ plus a bound on the eigenvalues of the deviation $\mathbf{P} - \mathbf{P}_0$:

$$\mathcal{E} = \{\mathbf{P} \in \mathbb{S}^n : -\gamma \mathbf{I} \preceq \mathbf{P} - \mathbf{P}_0 \preceq \gamma \mathbf{I}\}$$

where $\gamma \in \mathbb{R}_+$ and $\mathbf{P}_0 \in \mathbb{S}^n_+$.

(c) An ellipsoid of matrices:

$$\mathcal{E} = \left\{ \mathbf{P}_0 + \sum_{i=1}^{K} \mathbf{P}_i u_i : ||\mathbf{u}||_2 \leq 1 \right\} .$$

You can assume $\mathbf{P}_i \in \mathbb{S}^n_+$, $i = 0, \ldots, K$.
Hint: A hyperbolic constraint of the form

$$||\mathbf{x}||_2^2 \leq yz\,,\ \ y \geq 0\,,\ \ z \geq 0\,,$$

where $\mathbf{x} \in \mathbb{R}^n$ and $y, z \in \mathbb{R}$ can be shown to be true if and only if the second-order cone (SOC) constraint

$$\left\| \begin{bmatrix} 2\mathbf{x} \\ y - z \end{bmatrix} \right\|_2 \leq y + z\,,\ \ y \geq 0\,, z \geq 0$$

holds true.

3. (10 points) (Adapted from CO-BV, Exercise 4.39) (*SDPs and congruence transformations:*) Consider the SDP

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & x_1\mathbf{F}_1 + \cdots + x_n\mathbf{F}_n + \mathbf{G} \preceq \mathbf{0} \end{array} ,$$

with $\mathbf{F}_i, \mathbf{G} \in \mathbb{S}^k$, $\mathbf{c} \in \mathbb{R}^n$.

(a) Suppose $\mathbf{R} \in \mathbb{R}^{k \times k}$ is nonsingular. Show that the SDP is equivalent to the SDP

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & x_1\widetilde{\mathbf{F}}_1 + \cdots + x_n\widetilde{\mathbf{F}}_n + \widetilde{\mathbf{G}} \preceq \mathbf{0} \end{array} ,$$

where $\widetilde{\mathbf{F}}_i = \mathbf{R}^T\mathbf{F}_i\mathbf{R}$, $\widetilde{\mathbf{G}} = \mathbf{R}^T\mathbf{G}\mathbf{R}$.

(b) Suppose there exists a nonsingular $\mathbf{R}$ such that $\widetilde{\mathbf{F}}_i$ and $\widetilde{\mathbf{G}}$ are diagonal. Show that the SDP is equivalent to an LP.

(c) Suppose there exists a nonsingular $\mathbf{R}$ such that $\widetilde{\mathbf{F}}_i$ and $\widetilde{\mathbf{G}}$ have the form

$$\widetilde{\mathbf{F}}_i = \begin{bmatrix} \alpha_i \mathbf{I} & \mathbf{a}_i \\ \mathbf{a}_i^T & \alpha_i \end{bmatrix}\,,\ i = 1, \ldots, n\,,\ \widetilde{\mathbf{G}} = \begin{bmatrix} \beta \mathbf{I} & \mathbf{b} \\ \mathbf{b}^T & \beta \end{bmatrix}\,,$$

where $\alpha_i, \beta \in \mathbb{R}$ and $\mathbf{a}_i, \mathbf{b} \in \mathbb{R}^{k-1}$. Show that the SDP is equivalent to an SOCP with a single SOC constraint.

4. (10 points) (Adapted from CO-BV, Exercise 5.12) (*Analytic centering:*) Derive a dual problem for

$$\text{minimize} \quad -\sum_{i=1}^{m} \log\bigl(b_i - \mathbf{a}_i^T \mathbf{x}\bigr) \ ,$$

with domain $\bigl\{\mathbf{x} : \mathbf{a}_i^T \mathbf{x} < b_i \, , \ i = 1, \ldots, m\bigr\}$. First introduce new variables y_i and equality constraints $y_i = b_i - \mathbf{a}_i^T \mathbf{x}$.

(The solution of this problem is called the *analytic center* of the linear inequalities $\mathbf{a}_i^T \mathbf{x} \leq b_i$, $i = 1, \ldots, m$. Analytic centers have geometric applications and play an important role in barrier methods used to numerically solve convex optimization problems.)

***5.** (30 points) (Adapted from CO-AE, Exercise 4.57) (*Capacity of a communication channel:*) We consider a communication channel with input $X(t) \in \{1, \ldots, n\}$ and output $Y(t) \in \{1, \ldots, m\}$, where t represents time. The relation between the input and output is given statistically:

$$p_{i,j} = \Pr\{Y(t) = i \,|\, X(t) = j\} \ , \ i = 1, \ldots, m \, , \ j = 1, \ldots, n \, .$$

The matrix $\mathbf{P} \in \mathbb{R}^{m \times n}$ is called the *channel transition matrix*, and the channel is called a *discrete memoryless channel* (since the inputs and outputs take on discrete values and the statistics do not vary with time here).

A famous result of Shannon states that information can be sent over the communication channel, with arbitrarily small probability of error, at any rate less than a number C, called the *channel capacity*, in bits per second. Shannon also showed that the capacity of a discrete memoryless channel can be found by solving an optimization problem. Assume that X has a probability distribution denoted $\mathbf{x} \in \mathbb{R}^n$, i.e.,

$$x_j = \Pr\{X = j\} \ , \ j = 1, \ldots, n \, .$$

The *mutual information* between X and Y is given by

$$I(X;Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_j p_{i,j} \log_2 \frac{p_{i,j}}{\sum_{k=1}^{n} x_k p_{i,k}} \, .$$

Then the channel capacity C is given by

$$C = \sup_{\mathbf{x}} I(X;Y) \ ,$$

where the supremum is over all possible probability distributions for the input X, i.e., over $\mathbf{x} \succeq \mathbf{0}$, $\mathbf{1}^T \mathbf{x} = 1$.

Show how the channel capacity can be computed using convex optimization.

Hint: Introduce the variable $\mathbf{y} = \mathbf{P}\mathbf{x}$, which gives the probability distribution of the output Y, and show that the mutual information can be expressed as

$$I(X;Y) = -\mathbf{c}^T \mathbf{x} - \sum_{i=1}^{m} y_i \log_2 y_i \ ,$$

where $c_j = -\sum_{i=1}^{m} p_{i,j} \log_2 p_{i,j} = \sum_{i=1}^{m} p_{i,j} \log_2 (1/p_{i,j})$, $j = 1, \ldots, n$.

Reading assignments:

1. Read through Chapter 4 and begin Chapter 5 of CO-BV.

Reminders:

Late homework policy for EE 150: Late homeworks will not be accepted. There will be no exceptions to this other than institute established emergency reasons, in which case a signed letter is required from an authorized official.

NCT Problems: Remember that problems with an asterisk, such as ***7** are no collaboration type (NCT) problems.

Texts: The abbreviation CO-BV corresponds to the textbook "*Convex Optimization*" by Stephen Boyd and Lieven Vandenberghe. In addition, CO-AE refers to the *Additional Exercises for Convex Optimization*, also by Boyd and Vandenberghe. Finally, CVX corresponds to the `cvx` *Users' Guide* by Michael Grant and Stephen Boyd.
