EE 150 - Applications of Convex Optimization in Signal Processing and Communications
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Homework Set #6

Due on **Friday, May 18** at 1 PM in 110 Moore.

1. (10 points) (Adapted from CO-BV, Exercise 5.1) (*Elementary example of duality:*) Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x-2)(x-4) \leq 0 \end{array},$$

with variable $x \in \mathbb{R}$.

(a) *Analysis of primal problem:* Give the feasible set, the optimal solution $x^\star$, and the optimal value $p^\star$.

(b) *Lagrangian and dual function:* Plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^\star \geq \inf_x \{L(x, \lambda)\}$ for $\lambda \geq 0$). Derive and sketch the Lagrange dual function g.

(c) *Lagrange dual problem:* State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal solution $\lambda^\star$ and the dual optimal value $d^\star$. Does strong duality hold?

(d) *Sensitivity analysis:* Let $p^\star(u)$ denote the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x-2)(x-4) \leq u \end{array},$$

as a function of the parameter u. Plot $p^\star(u)$. Verify that $\frac{dp^\star(0)}{du} = -\lambda^\star$.

2. (10 points) (Adapted from CO-BV, Exercise 5.19) (*Sum of the largest elements of a vector:*) Define $f : \mathbb{R}^n \to \mathbb{R}$ as

$$f(\mathbf{x}) \triangleq \sum_{i=1}^{r} x_{[i]},$$

where r is an integer between 1 and n, and $x_{[1]} \geq \cdots \geq x_{[r]}$ are the components of $\mathbf{x}$ sorted in decreasing order. In other words, $f(\mathbf{x})$ is the sum of the r largest elements of $\mathbf{x}$. In this problem, we study the constraint

$$f(\mathbf{x}) \leq \alpha.$$

As was shown in Lecture #7, this is a convex constraint, and equivalent to a set of $C(n, r) = n!/(r!(n-r)!)$ linear inequalities

$$x_{i_1} + \cdots + x_{i_r} \leq \alpha, \; 1 \leq i_1 < \cdots < i_r \leq n.$$

The purpose of this problem is to derive a more compact representation.

(a) Given a vector $\mathbf{x} \in \mathbb{R}^n$, show that $f(\mathbf{x})$ is equal to the optimal value of the LP

$$\begin{array}{ll} \text{maximize} & \mathbf{x}^T\mathbf{y} \\ \text{subject to} & \mathbf{0} \preceq \mathbf{y} \preceq \mathbf{1} \\ & \mathbf{1}^T\mathbf{y} = r \end{array},$$

with $\mathbf{y} \in \mathbb{R}^n$ as the variable.

(b) Derive the dual of the LP in part (a). Show that it can be written as

$$\begin{array}{ll} \text{minimize} & rt + \mathbf{1}^T\mathbf{u} \\ \text{subject to} & t\mathbf{1} + \mathbf{u} \succeq \mathbf{x} \\ & \mathbf{u} \succeq \mathbf{0} \end{array},$$

where the variables are $t \in \mathbb{R}$, $\mathbf{u} \in \mathbb{R}^n$. By duality, this LP has the same optimal value as the LP in (a), i.e., $f(\mathbf{x})$. We therefore have the following result: $\mathbf{x}$ satisfies $f(\mathbf{x}) \leq \alpha$ if and only if there exist $t \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^n$ such that

$$rt + \mathbf{1}^T\mathbf{u} \leq \alpha\,, \;\; t\mathbf{1} + \mathbf{u} \succeq \mathbf{x}\,, \;\; \mathbf{u} \succeq \mathbf{0}\,.$$

These conditions form a set of $2n+1$ linear inequalities in the $2n+1$ variables $\mathbf{x}$, $\mathbf{u}$, and t.

(c) As an application, consider the patch illumination example from Lecture #1. This can be shown to be expressed as the following SOCP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \mathbf{a}_k^T\mathbf{p} \leq I_{\text{des}}t\,, \;\; k = 1, \ldots, n \\ & \left\| \begin{bmatrix} 2\sqrt{I_{\text{des}}} \\ t - \mathbf{a}_k^T\mathbf{p} \end{bmatrix} \right\|_2 \leq t + \mathbf{a}_k^T\mathbf{p}\,, \;\; k = 1, \ldots, n \\ & \mathbf{0} \preceq \mathbf{p} \preceq p_{\max}\mathbf{1} \end{array}.$$

Here, the variables are $\mathbf{p} \in \mathbb{R}^m$ and $t \in \mathbb{R}$, while the problem data consists of $\mathbf{a}_k \in \mathbb{R}^m_+$ for $k = 1, \ldots, n$, $I_{\text{des}} \in \mathbb{R}_{++}$, and $p_{\max} \in \mathbb{R}_{++}$.
Suppose we add one of the constraints mentioned in Lecture #1 that no more than half of the total power is in any m_0 lamps. Show that with this additional constraint, the problem above can be formulated as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \mathbf{a}_k^T\mathbf{p} \leq I_{\text{des}}t\,, \;\; k = 1, \ldots, n \\ & \left\| \begin{bmatrix} 2\sqrt{I_{\text{des}}} \\ t - \mathbf{a}_k^T\mathbf{p} \end{bmatrix} \right\|_2 \leq t + \mathbf{a}_k^T\mathbf{p}\,, \;\; k = 1, \ldots, n \\ & \mathbf{0} \preceq \mathbf{p} \preceq p_{\max}\mathbf{1} \\ & m_0 s + \mathbf{1}^T\mathbf{q} \leq \frac{1}{2}\mathbf{1}^T\mathbf{p}\,, \;\; s\mathbf{1} + \mathbf{q} \succeq \mathbf{p}\,, \;\; \mathbf{q} \succeq \mathbf{0} \end{array},$$

with variables $\mathbf{t} \in \mathbb{R}$, $\mathbf{p} \in \mathbb{R}^m$, $s \in \mathbb{R}$, and $\mathbf{q} \in \mathbb{R}^m$. Note that this is also an SOCP.

3. (10 points) (Adapted from CO-BV, Exercise 5.20) (*Dual of the channel capacity problem:*) Derive a dual for the problem

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} + \sum_{i=1}^{m} y_i \log y_i \\ \text{subject to} & \mathbf{P}\mathbf{x} = \mathbf{y} \\ & \mathbf{x} \succeq \mathbf{0}\,,\ \mathbf{1}^T\mathbf{x} = 1 \end{array},$$

where $\mathbf{P} \in \mathbb{R}^{m\times n}$ has nonnegative elements, and its columns add up to unity (i.e., $\mathbf{P}^T\mathbf{1} = \mathbf{1}$). The variables are $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. (For $c_\ell = -\sum_{k=1}^{m} p_{k,\ell} \log p_{k,\ell}$, the optimal value is, up to a factor $\log 2$, the capacity of a discrete memoryless channel with channel transition probability matrix $\mathbf{P}$; see Problem 5 of Homework Set #5.)
Simplify the dual problem as much as possible.

4. (10 points) (Adapted from CO-AE, Exercise 4.14) (*Kantorovich inequality:*)

(a) Suppose $\mathbf{a} \in \mathbb{R}^n$ with $a_1 \geq \cdots \geq a_n > 0$, and $\mathbf{b} \in \mathbb{R}^n$ with $b_k = 1/a_k$.
Derive the KKT conditions for the convex optimization problem

$$\begin{array}{ll} \text{minimize} & -\log\big(\mathbf{a}^T\mathbf{x}\big) - \log\big(\mathbf{b}^T\mathbf{x}\big) \\ \text{subject to} & \mathbf{x} \succeq \mathbf{0}\,,\ \mathbf{1}^T\mathbf{x} = 1 \end{array}.$$

Show that $\mathbf{x} = (1/2, 0, \ldots, 0, 1/2)$ is optimal.

(b) Suppose $\mathbf{A} \in \mathbb{S}^n_{++}$ with eigenvalues λ_k sorted in decreasing order. Apply the result of part (a), with $a_k = \lambda_k$, to prove the *Kantorovich inequality*:

$$2\left(\mathbf{u}^T\mathbf{A}\mathbf{u}\right)^{1/2}\left(\mathbf{u}^T\mathbf{A}^{-1}\mathbf{u}\right)^{1/2} \leq \sqrt{\frac{\lambda_1}{\lambda_n}} + \sqrt{\frac{\lambda_n}{\lambda_1}}\,,$$

for all $\mathbf{u}$ with $||\mathbf{u}||_2 = 1$.

***5.** (30 points) (Adapted from CO-AE, Exercise 4.4) (*Source localization from range measurements:*) A signal emitted by a source at an unknown position $\mathbf{x} \in \mathbb{R}^n$ ($n = 2$ or $n = 3$) is received by m sensors at known positions $\mathbf{y}_1, \ldots, \mathbf{y}_m \in \mathbb{R}^n$. From the strength of the received signals, we can obtain noisy estimates d_k of the distances $||\mathbf{x} - \mathbf{y}_k||_2$. We are interested in estimating the source position $\mathbf{x}$ based on the measured distances d_k.
In the following problem, the error between the squares of the actual and observed distances is minimized:

$$\text{minimize} \quad f_0(\mathbf{x}) = \sum_{k=1}^{m}\left(||\mathbf{x} - \mathbf{y}_k||_2^2 - d_k^2\right)^2 .$$

Introducing a new variable $t = \mathbf{x}^T\mathbf{x}$, we can express this as

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{m}\left(t - 2\mathbf{y}_k^T\mathbf{x} + ||\mathbf{y}_k||_2^2 - d_k^2\right)^2 \\ \text{subject to} & \mathbf{x}^T\mathbf{x} - t = 0 \end{array}. \qquad (1)$$

The variables are $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Although this problem is not convex, it can be shown that strong duality holds. (It is a variation on the problem discussed on pg. 229 of CO-BV

and in Exercise 5.29 of CO-BV.)
Solve (1) for an example with $m = 5$,

$$\mathbf{y}_1 = \begin{bmatrix} 1.8 \\ 2.5 \end{bmatrix}, \ \mathbf{y}_2 = \begin{bmatrix} 2.0 \\ 1.7 \end{bmatrix}, \ \mathbf{y}_3 = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}, \ \mathbf{y}_4 = \begin{bmatrix} 1.5 \\ 2.0 \end{bmatrix}, \ \mathbf{y}_5 = \begin{bmatrix} 2.5 \\ 1.5 \end{bmatrix},$$

and

$$\mathbf{d} = (2.00, 1.24, 0.59, 1.31, 1.44)\ .$$

In Figure 1, some contour lines of the cost function f_0 are shown, along with the sensor positions $\mathbf{y}_k$ indicated by circles.

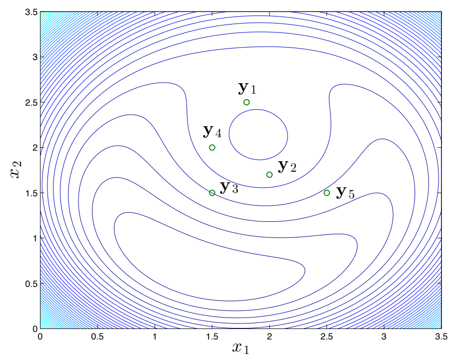

Figure 1: Contour plot of the objective $f_0(x_1, x_2)$ for the given problem data, with the sensor position vectors $\mathbf{y}_k$ indicated by circles.

To solve the problem, you can note that $\mathbf{x}^\star$ is easily obtained from the KKT conditions for (1) if the optimal multiplier $\nu^\star$ for the equality constraint is known. You can use one of the following two methods to find $\nu^\star$.

- Derive the dual problem, express it as an SDP, and solve it using `cvx`.
- Reduce the KKT conditions to a nonlinear equation in ν, and pick the correct solution.

Reading assignments:

1. Read through Chapter 5 and begin Chapter 6 of CO-BV.

Reminders:

Late homework policy for EE 150: Late homeworks will not be accepted. There will be no exceptions to this other than institute established emergency reasons, in which case a signed letter is required from an authorized official.

NCT Problems: Remember that problems with an asterisk, such as ***7** are no collaboration type (NCT) problems.

Texts: The abbreviation CO-BV corresponds to the textbook "*Convex Optimization*" by Stephen Boyd and Lieven Vandenberghe. In addition, CO-AE refers to the *Additional Exercises for Convex Optimization*, also by Boyd and Vandenberghe. Finally, CVX corresponds to the `cvx` *Users' Guide* by Michael Grant and Stephen Boyd.
