EE 150 - Applications of Convex Optimization in Signal Processing and Communications
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Homework Set #7

Due on **Friday, May 25** at 1 PM in 110 Moore.

1. (10 points) (Adapted from CO-AE, Exercise 5.3) (*Approximation with trigonometric polynomials:*) Suppose $y(t) : \mathbb{R} \to \mathbb{R}$ is a T-periodic function. We will approximate $y(t)$ with the trigonometric polynomial

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{K} a_k \cos\left(\frac{2\pi k t}{T}\right) + \sum_{k=1}^{K} b_k \sin\left(\frac{2\pi k t}{T}\right) .$$

We consider two approximations: one that minimizes the L_2-norm of the error, given by

$$||f - y||_2 = \left(\int_{-T/2}^{T/2} |f(t) - y(t)|^2 \, dt \right)^{1/2} ,$$

and one that minimizes the L_1-norm of the error, given by

$$||f - y||_1 = \int_{-T/2}^{T/2} |f(t) - y(t)| \, dt .$$

The L_2 approximation is, of course, given by the (truncated) Fourier expansion of y.
To find an L_1 approximation, we discretize t at $2N$ points given by

$$t_m = -\frac{T}{2} + \frac{mT}{2N} , \; m = 0, \ldots, 2N-1 ,$$

and approximate the L_1-norm as

$$||f - y||_1 \approx \frac{T}{2N} \sum_{m=0}^{2N-1} |f(t_m) - y(t_m)| .$$

(A standard rule of thumb is to take N to be at least 10 times larger than K.) The L_1 approximation (or really, an approximation of the L_1 approximation) can now be found using linear programming.
We consider a specific case, where $y(t)$ is a 1-periodic square-wave, defined for $-1/2 \leq t < 1/2$ as

$$y(t) = \begin{cases} 1 , & -1/4 \leq t < 1/4 \\ 0 , & \text{otherwise} \end{cases} .$$

(The graph of y over a few cycles explains the name 'square-wave'.)
Find the optimal L_2 approximation and optimal (discretized) L_1 approximation for $K = 10$. You can find the optimal L_2 approximation analytically, or by solving a least-squares problem associated with the discretized version of the problem. Since y is even, you can take the sine coefficients in your approximations to be zero. Show y and the two approximations on the same plot.
In addition, plot a histogram of the residuals (i.e., the numbers $f(t_m) - y(t_m)$) for the two approximations. Use the same axis range, so the two residual distributions can be easily compared. (The MATLAB command `hist` might be helpful here.) Make some brief comments about what you see.

2. (10 points) (Adapted from CO-AE, Exercise 5.6) (*Total variation image interpolation:*) A grayscale image is represented as an $m \times n$ matrix of intensities $\mathbf{U}^{\text{orig}}$. You are given the values $U^{\text{orig}}_{k,\ell}$, for $(k,\ell) \in \mathcal{K}$, where $\mathcal{K} \subset \{1,\ldots,m\} \times \{1,\ldots,n\}$. Your job is to *interpolate* the image, by guessing the missing values. The reconstructed image will be represented by $\mathbf{U} \in \mathbb{R}^{m\times n}$, where $\mathbf{U}$ satisfies the interpolation conditions $U_{k,\ell} = U^{\text{orig}}_{k,\ell}$ for $(k,\ell) \in \mathcal{K}$.
The reconstruction is found by minimizing a roughness measure subject to the interpolation conditions. One common roughness measure is the ℓ_2 variation (squared), given by

$$\sum_{k=2}^{m}\sum_{\ell=2}^{n}\left((U_{k,\ell}-U_{k-1,\ell})^2+(U_{k,\ell}-U_{k,\ell-1})^2\right) .$$

Another method minimizes instead the *total variation*, given by

$$\sum_{k=2}^{m}\sum_{\ell=2}^{n}\left(|U_{k,\ell}-U_{k-1,\ell}|+|U_{k,\ell}-U_{k,\ell-1}|\right) .$$

Evidently, both methods lead to convex optimization problems.
Carry out ℓ_2 and total variation interpolation on the problem instance with data given in `tv_img_interp.m` obtained from the image `tv_img_interp.png`. This will define `m`, `n`, and matrices `Uorig` and `Known`. The matrix `Known` is $m \times n$, with (k,ℓ)-th entry one if $(k,\ell) \in \mathcal{K}$, and zero otherwise. The `.m` file also has skeleton plotting code. (We give you the entire original image so you can compare your reconstruction to the original; obviously your solution cannot access $U^{\text{orig}}_{k,\ell}$ for $(k,\ell) \notin \mathcal{K}$.)

3. (10 points) (Adapted from CO-AE, Exercise 5.10) (*Identifying a sparse linear dynamical system:*) A linear dynamical system has the form

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{w}(t) \ , \ t = 1,\ldots,T-1 \, ,$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state, $\mathbf{u}(t) \in \mathbb{R}^m$ is the input signal, and $\mathbf{w}(t) \in \mathbb{R}^n$ is the process noise, each at time t. We assume the process noises are independent and identically distributed (i.i.d.) with distribution $\mathcal{N}(\mathbf{0},\mathbf{W})$, where $\mathbf{W} \succ \mathbf{0}$ is the covariance matrix. The matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is called the dynamics matrix or the state transition matrix, and the matrix $\mathbf{B} \in \mathbb{R}^{n\times m}$ is called the input matrix.
You are given accurate measurements of the state and input signal, i.e., $\mathbf{x}(1),\ldots,\mathbf{x}(T)$ and $\mathbf{u}(1),\ldots,\mathbf{u}(T-1)$, and $\mathbf{W}$ is known. Your job is to find a state transition matrix $\widehat{\mathbf{A}}$ and input matrix $\widehat{\mathbf{B}}$ from this set of data, that are plausible, and in addition are sparse, i.e., have many zero entries. (The sparser the better.)
By doing this, you are effectively estimating the structure of the dynamical system, i.e., you are determining which components of $\mathbf{x}(t)$ and $\mathbf{u}(t)$ affect which components of $\mathbf{x}(t+1)$. In some applications, this structure might be more interesting than the actual values of the (nonzero) coefficients in $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$.
By plausible, we mean that

$$\sum_{t=1}^{T-1}\left|\left|\mathbf{W}^{-1/2}\left(\mathbf{x}(t+1)-\widehat{\mathbf{A}}\mathbf{x}(t)-\widehat{\mathbf{B}}\mathbf{u}(t)\right)\right|\right|_2^2 \le n\,(T-1) + 2\sqrt{2n\,(T-1)}\,.$$

(You can just take this as our definition of plausible. But to explain this choice, we note that when $\widehat{\mathbf{A}} = \mathbf{A}$ and $\widehat{\mathbf{B}} = \mathbf{B}$, the left-hand side (LHS) is χ^2, with $n\,(T-1)$ degrees of freedom, and so has mean $n\,(T-1)$ and standard deviation $\sqrt{2n\,(T-1)}$. Thus, the constraint above states that the LHS does not exceed the mean by more than 2 standard deviations.)

(a) Describe a method for finding $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$, based on convex optimization.
We are looking for a *very simple* method, that involves solving *one* convex optimization problem. (There are many extensions of this basic method, that would improve the simple method, i.e., yield sparser $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$ that are still plausible. We are not asking you to describe or implement any of these.)

(b) Carry out your method on the data found in `sparse_lds_data.m`. Give the values of $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$ that you find, and verify that they are plausible.
In the data file, we give you the true values of $\mathbf{A}$ and $\mathbf{B}$, so you can evaluate the performance of your method. (Needless to say, you are not allowed to use these values when forming $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$.) Using these true values, give the number of false positives and false negatives in both $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$. A false positive in $\widehat{\mathbf{A}}$, for example, is an entry that is nonzero, while the corresponding entry in $\mathbf{A}$ is zero. Similarly, a false negative is an entry of $\widehat{\mathbf{A}}$ that is zero, while the corresponding entry of $\mathbf{A}$ is nonzero. To judge whether an entry of $\widehat{\mathbf{A}}$ (or $\widehat{\mathbf{B}}$) is nonzero, you can use the test $\left|\widehat{A}_{k,\ell}\right| \geq 0.01$ (or $\left|\widehat{B}_{k,\ell}\right| \geq 0.01$).

4. (10 points) (Adapted from CO-AE, Exercise 5.14) (*Spectrum analysis with quantized measurements:*) A sample is made up of n compounds, in quantities $q_i \geq 0$, for $i = 1, \ldots, n$. Each compound has a (nonnegative) spectrum, which we represent as a vector $\mathbf{s}^{(i)} \in \mathbb{R}^m_+$, for $i = 1, \ldots, n$. (Precisely what $\mathbf{s}^{(i)}$ means does not matter to us here.) The spectrum of the sample is given by $\mathbf{s} = \sum_{i=1}^n q_i \mathbf{s}^{(i)}$. We can write this more compactly as $\mathbf{s} = \mathbf{S}\mathbf{q}$, where $\mathbf{S} \in \mathbb{R}^{m \times n}_+$ is a matrix whose columns are $\mathbf{s}^{(1)}, \ldots, \mathbf{s}^{(n)}$.
Measurement of the spectrum of the sample gives us an interval for each spectrum value, i.e., $\mathbf{l}, \mathbf{u} \in \mathbb{R}^m_+$ for which

$$l_i \leq s_i \leq u_i \,, \ i = 1, \ldots, m \,.$$

(In other words, we do not get $\mathbf{s}$ directly.) This occurs, for example, if our measurements are quantized.
Given $\mathbf{l}$ and $\mathbf{u}$ (and $\mathbf{S}$), we cannot in general deduce $\mathbf{q}$ exactly. Instead, we ask you to do the following. For each compound i, find the range of possible values for q_i that are consistent with the spectrum measurements. We will denote these ranges as $q_i \in \left[q_i^{\min}, q_i^{\max}\right]$. Your job is to find $q_i^{\min}$ and $q_i^{\max}$.
Note that if $q_i^{\min}$ is large, we can confidently conclude that there is a significant amount of compound i in the sample. If $q_i^{\max}$ is small, we can confidently conclude that there is not much of compound i in the sample.

(a) Explain how to find $q_i^{\min}$ and $q_i^{\max}$, given $\mathbf{S}$, $\mathbf{l}$, and $\mathbf{u}$.

(b) Carry out the method of part (a) for the problem instance given in `spectrum_data.m`. (Executing this file defines the problem data, and plots the compound spectra and measurement bounds.) Plot the minimum and maximum values versus i, using the commented out code in the data file. Report your values for $q_4^{\min}$ and $q_4^{\max}$.

***5.** (30 points) (Adapted from CO-AE, Exercise 12.6) (*Antenna array weight design:*) We consider an array of n omnidirectional antennas in a plane, at positions (x_k, y_k), for $k = 1, \ldots, n$. A narrowband signal plane wave transmitted at a carrier frequency of F_c is incident from an angle θ, as shown in Figure 1. The delay of the signal at the k-th antenna relative to the origin induces a phase shift of the complex baseband signal received at the

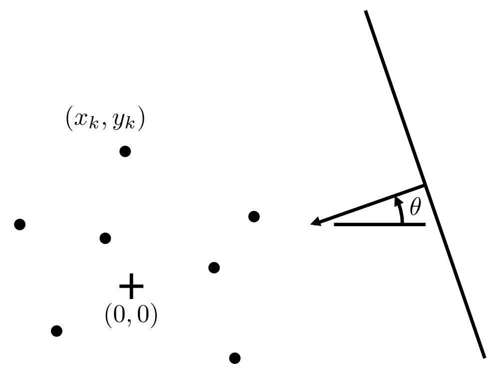

Figure 1: Pictorial view of a plane wave incident upon an antenna array with angle θ.

k-th antenna of the form $e^{j\phi_k(\theta)} \triangleq \exp\left(j\left(\frac{2\pi x_k}{\lambda}\cos\theta + \frac{2\pi y_k}{\lambda}\sin\theta\right)\right)$, where $\lambda = c/F_{\mathrm{c}}$ is the wavelength and c is the speed of light. Defining the dimensionless normalized antenna positions $\overline{x}_k \triangleq \frac{2\pi x_k}{\lambda}$ and $\overline{y}_k \triangleq \frac{2\pi y_k}{\lambda}$, we have $\phi_k(\theta) = (\overline{x}_k \cos\theta + \overline{y}_k \sin\theta)$. The complex baseband signals of the n antennas are combined linearly to form the output of the antenna array given by the following.

$$
\begin{aligned}
G(\theta) &= \sum_{k=1}^{n} w_k e^{j\phi_k(\theta)} \,, \\
&= \sum_{k=1}^{n} \left[\left(w_{\mathrm{re},k}\cos\phi_k(\theta) - w_{\mathrm{im},k}\sin\phi_k(\theta)\right) + j\left(w_{\mathrm{re},k}\sin\phi_k(\theta) + w_{\mathrm{im},k}\cos\phi_k(\theta)\right)\right] \,,
\end{aligned}
$$

where $w_{\mathrm{re},k} \triangleq \mathrm{Re}[w_k]$ and $w_{\mathrm{im},k} \triangleq \mathrm{Im}[w_k]$ for $k = 1, \ldots, n$. The complex weights in the above linear combination are called the *antenna array coefficients* or *shading coefficients*, and will be the design variables in the problem. For a given set of weights, the combined output $G(\theta)$ is a function of the angle of arrival θ of the plane wave. The design problem is to select weights w_k that achieve a desired directional pattern $G(\theta)$.

We now describe a basic weight design problem. To steer the beam in a desired direction, we require unit gain in a target direction θ^{tar}, i.e., $G\left(\theta^{\mathrm{tar}}\right) = 1$. In addition, we want $|G(\theta)|$ small for $\left|\theta - \theta^{\mathrm{tar}}\right| \geq \Delta$, where 2Δ is our beamwidth. To do this, we can minimize

$$
\max_{|\theta - \theta^{\mathrm{tar}}| \geq \Delta} |G(\theta)| \,,
$$

where the maximum is over all $\theta \in [-\pi, \pi)$ with $\left|\theta - \theta^{\mathrm{tar}}\right| \geq \Delta$. This number is called the sidelobe level for the array and our goal is to minimize it. If we achieve a small sidelobe level, then the array is relatively insensitive to signals arriving from directions that are more than Δ away from the target direction. This results in the optimization problem

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\max_{|\theta - \theta^{\mathrm{tar}}| \geq \Delta} |G(\theta)| \\
\text{subject to} & G\left(\theta^{\mathrm{tar}}\right) = 1
\end{array} \,,
$$

with $\mathbf{w} \in \mathbb{C}^n$ as the variable, where $[\mathbf{w}]_k = w_k$.

The objective function can be approximated by first discretizing the angle of arrival θ with, say, N values $\theta_1, \ldots, \theta_N$ (assumed, for example, uniformly spaced) over the interval $\left[\theta^{\mathrm{tar}} + \Delta, \theta^{\mathrm{tar}} + 2\pi - \Delta\right]$. Then, we replace the original objective function with

$$
\max\left\{ |G(\theta_\ell)| : \left|\theta_\ell - \theta^{\mathrm{tar}}\right| \geq \Delta \,,\ \ell = 1, \ldots, N \right\} \,.
$$

(a) Formulate the antenna array weight design problem as an SOCP.

(b) Solve an instance using `cvx`, with $n = 40$, $\lambda = 2\pi$, $\theta^{\text{tar}} = 15^\circ$, $\Delta = 15^\circ$, $N = 400$, and antenna positions generated using

```
>> rand('state',0);
>> n = 40;
>> x = 30 * rand(n,1);
>> y = 30 * rand(n,1);
```

Compute the optimal weights and make a plot of $|G(\theta)|$ in decibels (dB) versus θ (i.e., plot $20\log_{10}(|G(\theta)|)$ as a function of θ).
Hint: `cvx` can directly handle complex variables, and recognizes the modulus `abs(x)` of a complex number as a convex function of its real and imaginary parts, so you do not need to explicitly form the SOCP from part (a).

Reading assignments:

1. Read through Chapter 6 and begin Chapter 7 of CO-BV. Look over parts of CVX as necessary.

Reminders:

Late homework policy for EE 150: Late homeworks will not be accepted. There will be no exceptions to this other than institute established emergency reasons, in which case a signed letter is required from an authorized official.

NCT Problems: Remember that problems with an asterisk, such as ***7** are no collaboration type (NCT) problems.

Texts: The abbreviation CO-BV corresponds to the textbook "*Convex Optimization*" by Stephen Boyd and Lieven Vandenberghe. In addition, CO-AE refers to the *Additional Exercises for Convex Optimization*, also by Boyd and Vandenberghe. Finally, CVX corresponds to the `cvx` *Users' Guide* by Michael Grant and Stephen Boyd.
