EE 150 - Applications of Convex Optimization in Signal Processing and Communications
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Homework Set #8

Due on **Friday, June 8** at 1 PM in 110 Moore.

1. (10 points) (Adapted from CO-AE, Exercise 12.1) (*FIR low-pass filter design:*) Consider a finite impulse response (FIR) filter with impulse response $h[n]$, which is only nonzero for $n = 0, \ldots, N$, where N is some even integer. Suppose that $h[n]$ is symmetric, i.e., $h[n] = h[N-n]$ for all n. Then, if $h[n]$ is also real, it can be shown that the frequency response $H\left(e^{j2\pi f}\right)$ is given by

$$H\left(e^{j2\pi f}\right) = e^{-j\pi N f} H_R(f) \ , \tag{1}$$

where $H_R(f)$ is the amplitude response, or zero-phase response, which is given by

$$H_R(f) = \sum_{k=0}^{M} b_k \cos(2\pi k f) \ ,$$

where $M \triangleq N/2$ and $b_k \in \mathbb{R}$ for $k = 0, \ldots, M$. Since the phase term from (1) is linear, the filter $h[n]$ is said to be a *linear phase* filter.

In this problem, we will design a filter to meet certain frequency band specifications. The design variable here is the vector $\mathbf{b} = \begin{bmatrix} b_0 \cdots b_M \end{bmatrix}^T \in \mathbb{R}^{M+1}$. We will explore the design of a low-pass filter. Specifically, we will set specifications on the magnitude response $\left|H\left(e^{j2\pi f}\right)\right|$ in dB (i.e., $20\log_{10}\left(\left|H\left(e^{j2\pi f}\right)\right|\right)$), as shown in Figure 1(a), map those specifications to equivalent ones for the amplitude response $H_R(f)$, as shown in Figure 1(b), and then select the filter coefficient vector $\mathbf{b}$ to accommodate this latter set of specifications.

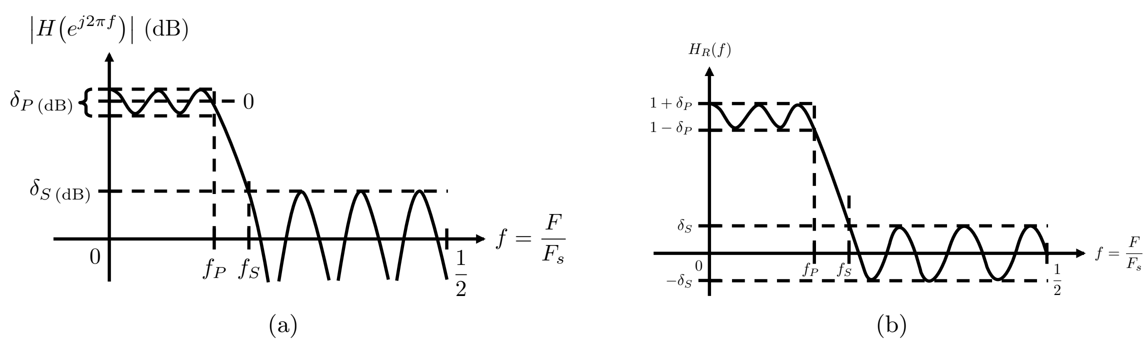

Figure 1: Low-pass filter responses: (a) magnitude response $\left|H\left(e^{j2\pi f}\right)\right|$ in dB (i.e., $20\log_{10}\left(\left|H\left(e^{j2\pi f}\right)\right|\right)$) and (b) amplitude response $H_R(f)$.

Referring to Figure 1(a), the filter is characterized by a *passband frequency* f_P, a *stopband frequency* f_S, a *peak-to-peak passband ripple* $\delta_{P\,(\mathrm{dB})}$, and a *stopband ripple level* $\delta_{S\,(\mathrm{dB})}$. The ripple specifications can be mapped to a linear scale, which is then used to design the amplitude response $H_R(f)$. This leads to a *passband ripple level* δ_P and a *stopband ripple level* δ_S. The relation between these specifications is given as follows.

$$\delta_P = \frac{K-1}{K+1} \ , \ \text{where } K = 10^{\delta_{P\,(\mathrm{dB})}/20} \ , \ \delta_S = 10^{\delta_{S\,(\mathrm{dB})}/20} \ .$$

The goal is to design the filter to meet the following specifications.

- For $0 \leq f \leq f_P$, we want $1 - \delta_P \leq H_R(f) \leq 1 + \delta_P$. This is the passband requirement.
- For $f_S \leq f \leq 1/2$, we want $-\delta_S \leq H_R(f) \leq \delta_S$. This is the stopband requirement.

For parts (a)-(c), explain how to formulate the given design problem (for $H_R(f)$) as a convex or quasiconvex optimization problem.

(a) *Maximum stopband attenuation:* Suppose we fix f_P, δ_P, f_S, and M, and wish to maximize the stopband attenuation, i.e., minimize δ_S.

(b) *Minimum transition band:* Suppose we fix f_P, δ_P, δ_S, and M, and wish to minimize f_S, i.e., we wish to minimize the 'transition' band between f_P and f_S with f_P fixed.

(c) *Shortest length filter:* Suppose we fix f_P, δ_P, f_S, and δ_S, and we wish to find the smallest M that can meet the specifications. As the filter length is $N + 1 = 2M + 1$, this is tantamount to finding the shortest length FIR filter that can meet the specifications.

(d) *Numerical filter design:* Use `cvx` to find the shortest length filter that satisfies the following filter specifications.

$$f_P = \frac{1}{6}\,, \quad \delta_{P\,(\mathrm{dB})} = 0.1\,, \quad f_S = \frac{1}{5}\,, \quad \delta_{S\,(\mathrm{dB})} = -30\,.$$

For this subproblem, you may discretize the constraints in frequency, which means the following. Choose L large (say, 500; an old rule of thumb is that L should be at least $15M$), and set $f_\ell = \ell/(2L)$, where $\ell = 0, \ldots, L$. Then replace the specifications with

- For ℓ with $0 \leq f_\ell \leq f_P$, use $1 - \delta_P \leq H_R(f_\ell) \leq 1 + \delta_P$.
- For ℓ with $f_S \leq f_\ell \leq \frac{1}{2}$, use $-\delta_S \leq H_R(f_\ell) \leq \delta_S$.

Report the optimal value of M. Then, plot the magnitude response of the filter in dB, i.e., $20\log_{10}\left(\left|H\left(e^{j2\pi f}\right)\right|\right)$, and verify visually that it meets the desired constraints.

2. (10 points) (Adapted from CO-AE, Exercise 7.8) (*Bounding object position from multiple camera views:*) A small object is located at an unknown position in $\mathbf{x} \in \mathbb{R}^3$, and viewed by a set of m cameras. Our goal is to find a box in $\mathbb{R}^3$,

$$\mathcal{B} = \left\{\mathbf{z} \in \mathbb{R}^3 : \mathbf{l} \preceq \mathbf{z} \preceq \mathbf{u}\right\}\,,$$

for which we can guarantee $\mathbf{x} \in \mathcal{B}$. We want the smallest possible such bounding box. (Although it does not matter, we can use volume to judge the 'smallest' among the boxes.)

Now we describe the cameras. The object at location $\mathbf{x} \in \mathbb{R}^3$ creates an image on the image plane of camera i at location

$$\mathbf{v}_i = \frac{1}{\mathbf{c}_i^T\mathbf{x} + d_i}\left(\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\right) \in \mathbb{R}^2\,.$$

The matrices $\mathbf{A}_i \in \mathbb{R}^{2\times 3}$, vectors $\mathbf{b}_i \in \mathbb{R}^2$ and $\mathbf{c}_i \in \mathbb{R}^3$, and scalars $d_i \in \mathbb{R}$ are known, and depend on the camera positions and orientations. We assume that $\mathbf{c}_i^T\mathbf{x} + d_i > 0$. The 3×4 matrix

$$\mathbf{P}_i = \begin{bmatrix} \mathbf{A}_i & \mathbf{b}_i \\ \mathbf{c}_i^T & d_i \end{bmatrix}$$

is called the *camera matrix* (for camera i). It is often (but not always) the case that the first 3 columns of $\mathbf{P}_i$ (i.e., $\mathbf{A}_i$ stacked above $\mathbf{c}_i^T$) form an orthogonal matrix, in which case the camera is called *orthographic*.

We do not have direct access to the image point $\mathbf{v}_i$; we only know the (square) pixel that it lies in. In other words, the camera gives us a measurement $\widehat{\mathbf{v}}_i$ (the center of the pixel that the image point lies in); we are guaranteed that

$$||\mathbf{v}_i - \widehat{\mathbf{v}}_i||_\infty \leq \rho_i/2\,,$$

where ρ_i is the pixel width (and height) of camera i. (We know nothing else about $\mathbf{v}_i$; it could be any point in this pixel.)

Given the data $\mathbf{A}_i$, $\mathbf{b}_i$, $\mathbf{c}_i$, d_i, $\widehat{\mathbf{v}}_i$, and ρ_i, we are to find the smallest box $\mathcal{B}$ (i.e., find the vectors $\mathbf{l}$ and $\mathbf{u}$) that is guaranteed to contain $\mathbf{x}$. In other words, find the smallest box in $\mathbb{R}^3$ that contains all points consistent with the observations from the camera.

(a) Explain how to solve this using convex or quasiconvex optimization. You must explain any transformations you use, any new variables you introduce, etc. If the convexity or quasiconvexity of any function in your formulation is not obvious, be sure to justify it.

(b) Solve the specific problem instance given in the file `camera_data.m`. Be sure that your final numerical answer (i.e., $\mathbf{l}$ and $\mathbf{u}$) stands out.

3. (10 points) (Adapted from CO-AE, Exercise 6.5) (*Estimating a vector with an unknown measurement nonlinearity:*) We want to estimate a vector $\mathbf{x} \in \mathbb{R}^n$, given some measurements

$$y_i = \phi\big(\mathbf{a}_i^T\mathbf{x} + v_i\big)\ ,\ i = 1, \ldots, m\,.$$

Here, $\mathbf{a}_i \in \mathbb{R}^n$ are known, v_i are independent and identically distributed (i.i.d.) $\mathcal{N}\big(0, \sigma^2\big)$ random noises, and $\phi : \mathbb{R} \to \mathbb{R}$ is an unknown monotonic increasing function, known to satisfy

$$\alpha \leq \phi'(u) \leq \beta\,,$$

for all u. (Here, α and β are known positive constants, with $\alpha < \beta$.) We want to find a maximum likelihood (ML) estimate of $\mathbf{x}$ and ϕ, given y_i. (We also know $\mathbf{a}_i$, σ, α, and β.)

This sounds like an infinite-dimensional problems, since one of the parameters we are trying to estimate is a function. In fact, we only need to know the m numbers $z_i = \phi^{-1}(y_i)$ for $i = 1, \ldots, m$. So by estimating ϕ, we really mean estimating the m numbers $z_1, \ldots, z_m$. (These numbers are not arbitrary; they must be consistent with the prior information $\alpha \leq \phi'(u) \leq \beta$ for all u).

(a) Explain how to find an ML estimate of $\mathbf{x}$ and ϕ (i.e., $z_1, \ldots, z_m$) using convex optimization.

(b) Carry out your method on the data given in `nonlin_meas_data.m`, which includes a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, with rows $\mathbf{a}_1^T, \ldots, \mathbf{a}_m^T$. Give $\widehat{\mathbf{x}}_{\text{ml}}$, the ML estimate of $\mathbf{x}$. Plot your estimated function $\widehat{\phi}_{\text{ml}}$. (You can do this by plotting $[\widehat{\mathbf{z}}_{\text{ml}}]_i$ versus y_i, with y_i on the vertical axis and $[\widehat{\mathbf{z}}_{\text{ml}}]_i$ on the horizontal axis.)

Hint: You can assume the measurements are numbered so that y_i are sorted in nondecreasing order, i.e., $y_1 \leq \cdots \leq y_m$. (The data given in the problem instance for part (b) is given in this order.)

4. (10 points) (Adapted from CO-AE, Exercise 7.10) (*Ellipsoidal peeling:*) In this exercise, you will implement an outlier identification and removal technique called *ellipsoidal peeling.* We are given a set of points $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^n$; our goal is to find a small (measured by volume) ellipsoid $\mathcal{E}$, for which $\mathbf{x}_i \in \mathcal{E}$ for $i \notin \mathcal{O}$, where $\mathcal{O} \subset \{1, \ldots, N\}$ is the set of 'outliers'. Of course, there is a trade-off between $\text{card}(\mathcal{O})$ (the cardinality of the set of outliers) and $\text{vol}(\mathcal{E})$. Once we choose $\mathcal{O}$, though, we can find $\mathcal{E}$ as the minimum volume ellipsoid that contains $\mathbf{x}_i$ for $i \notin \mathcal{O}$.

Ellipsoidal peeling is a heuristic for finding reasonable choices for $\mathcal{O}$. We start with $\mathcal{O} = \varnothing$, and find the minimum volume ellipsoid $\mathcal{E}$ containing all $\mathbf{x}_i$ for $i \notin \mathcal{O}$ (which, at this step, is all $\mathbf{x}_i$). Some of the points $\mathbf{x}_i$ will be on the surface of $\mathcal{E}$; we add these points to $\mathcal{O}$, and repeat. Roughly speaking, in each step, we 'peel off' the points that lie on the surface of the smallest volume enclosing ellipsoid. We then plot $\text{vol}(\mathcal{E})$ versus $\text{card}(\mathcal{O})$, and hope that we see a clear knee of the curve.

There are many variations on this approach. For example, instead of dropping all points on the surface of the current ellipsoid, we might drop only the one that corresponds to the largest Lagrange multiplier for the constraint that requires $\mathbf{x}_i \in \mathcal{E}$.

Apply ellipsoidal peeling to the data given in `ellip_peel_data.m`. Specifically, use the approach in which the point corresponding to the largest Lagrange multiplier is removed at each stage. Carry this out a total of 30 times and plot $\text{vol}(\mathcal{E})$ (on a log scale) versus $\text{card}(\mathcal{O})$. Based on this plot, estimate the number of outliers present in the data. Compare this number to that obtained using the 'eyeball' test in which you estimate the number of outliers present by plotting the data. This should bring to light the advantages and disadvantages of the ellipsoidal peeling algorithm for cases in which the data cannot be visualized.

Hint: In `cvx`, you should use `det_rootn` (which is handled exactly), rather than `log_det` (which is handled using an inefficient iterative procedure).

***5.** (30 points) (Adapted from CO-AE, Exercise 4.10) (*Binary least-squares:*) We consider the non-convex least-squares approximation problem with binary constraints

$$\begin{array}{ll} \text{minimize} & ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2 \\ \text{subject to} & x_k^2 = 1\,,\ k = 1, \ldots, n \end{array}\,, \tag{2}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. We assume that $\text{rank}(\mathbf{A}) = n$, i.e., $\mathbf{A}^T\mathbf{A}$ is nonsingular.

One possible application of this problem is as follows. A signal $\widehat{\mathbf{x}} \in \{-1, 1\}^n$ is sent over a noisy channel and received as $\mathbf{b} = \mathbf{A}\widehat{\mathbf{x}} + \mathbf{v}$, where $\mathbf{v} \sim \mathcal{N}\big(\mathbf{0}_{m\times 1}, \sigma^2\mathbf{I}_m\big)$ is Gaussian noise. The solution of (2) is the ML estimate of the input signal $\widehat{\mathbf{x}}$, based on the received signal $\mathbf{b}$.

(a) Derive the Lagrange dual of (2) and express it as an SDP.
Hint: You may want to consider a range condition for $\big(-\mathbf{A}^T\mathbf{b}\big)$ rather than the more obvious choice $\mathbf{A}^T\mathbf{b}$.

(b) Derive the dual of the SDP in part (a) and show that it is equivalent to

$$\begin{array}{ll} \text{minimize} & \text{tr}\big(\mathbf{A}^T\mathbf{A}\mathbf{Z}\big) - 2\mathbf{b}^T\mathbf{A}\mathbf{z} + \mathbf{b}^T\mathbf{b} \\ \text{subject to} & \text{diag}(\mathbf{Z}) = \mathbf{1} \\ & \begin{bmatrix} \mathbf{Z} & \mathbf{z} \\ \mathbf{z}^T & 1 \end{bmatrix} \succeq \mathbf{0} \end{array}\,. \tag{3}$$

Interpret this problem as a relaxation of (2). Show that if

$$\operatorname{rank}\left(\begin{bmatrix} \mathbf{Z} & \mathbf{z} \\ \mathbf{z}^T & 1 \end{bmatrix}\right) = 1 \tag{4}$$

at the optimum of (3), then the relaxation is exact, i.e., the optimal values of problems (2) and (3) are equal, and the optimal solution $\mathbf{z}$ of (3) is optimal for (2). This suggests a heuristic for rounding the solution of the SDP (3) to a feasible solution of (2), if (4) does not hold. We compute the eigenvalue decomposition

$$\begin{bmatrix} \mathbf{Z} & \mathbf{z} \\ \mathbf{z}^T & 1 \end{bmatrix} = \sum_{i=1}^{n+1} \lambda_i \begin{bmatrix} \mathbf{v}_i \\ t_i \end{bmatrix} \begin{bmatrix} \mathbf{v}_i \\ t_i \end{bmatrix}^T,$$

where $\mathbf{v}_i \in \mathbb{R}^n$ and $t_i \in \mathbb{R}$, and approximate the matrix by a rank-one matrix

$$\begin{bmatrix} \mathbf{Z} & \mathbf{z} \\ \mathbf{z}^T & 1 \end{bmatrix} \approx \lambda_1 \begin{bmatrix} \mathbf{v}_1 \\ t_1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ t_1 \end{bmatrix}^T.$$

(Here, we assume the eigenvalues are sorted in decreasing order.) Then, we take $\mathbf{x} = \operatorname{sgn}(\mathbf{v}_1)$ as our guess of a good solution of (2).

Hint: To handle the linear matrix inequality constraint (LMI) appearing in the SDP, consider a Lagrange multiplier matrix of the form

$$\begin{bmatrix} \mathbf{Z} & \mathbf{z} \\ \mathbf{z}^T & \lambda \end{bmatrix},$$

where $\mathbf{Z} \in \mathbb{S}^n$, $\mathbf{z} \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$.

(c) We can also give a probabilistic interpretation of the relaxation (3). Suppose we interpret $\mathbf{z}$ and $\mathbf{Z}$ as the first and second moments of a random variable $\mathbf{v} \in \mathbb{R}^n$ (i.e., $\mathbf{z} = E[\mathbf{v}]$ and $\mathbf{Z} = E\left[\mathbf{v}\mathbf{v}^T\right]$). Show that (3) is equivalent to the problem

$$\begin{array}{ll} \text{minimize} & E\left[\|\mathbf{A}\mathbf{v} - \mathbf{b}\|_2^2\right] \\ \text{subject to} & E\left[v_k^2\right] = 1\,,\ k = 1, \ldots, n \end{array},$$

where we minimize over all possible probability distributions of $\mathbf{v}$.

This interpretation suggests another heuristic method for computing suboptimal solutions of (2) based on the result of (3). We choose a distribution with first and second moments $E[\mathbf{v}] = \mathbf{z}$ and $E\left[\mathbf{v}\mathbf{v}^T\right] = \mathbf{Z}$, respectively (for example, the Gaussian distribution $\mathcal{N}\left(\mathbf{z}, \mathbf{Z} - \mathbf{z}\mathbf{z}^T\right)$). We generate a number of samples $\widetilde{\mathbf{v}}$ from the distribution and round them to feasible solutions $\mathbf{x} = \operatorname{sgn}(\widetilde{\mathbf{v}})$. We keep the solution with the lowest objective value as our guess of the optimal solution of (2).

(d) Solve the dual problem (3) using `cvx`. Generate problem instances using the MATLAB code

```
randn('state',0)
m = 50;
n = 40;
A = randn(m,n);
xhat = sign(randn(n,1));
b = A*xhat + s*randn(m,1);
```

for four values of the noise level $\mathtt{s}$: $\mathtt{s} = 0.5$, $\mathtt{s} = 1$, $\mathtt{s} = 2$, $\mathtt{s} = 3$. For each problem instance, compute suboptimal feasible solutions $\mathbf{x}$ using the following heuristics and compare the results.

(i) $\mathbf{x}^{(\mathrm{a})} = \mathrm{sgn}(\mathbf{x}_{\mathrm{ls}})$, where $\mathbf{x}_{\mathrm{ls}}$ is the solution of the least-squares problem

$$\text{minimize} \quad ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2 \ .$$

(ii) $\mathbf{x}^{(\mathrm{b})} = \mathrm{sgn}(\mathbf{z})$, where $\mathbf{z}$ is the optimal value of the variable $\mathbf{z}$ in the SDP (3).

(iii) $\mathbf{x}^{(\mathrm{c})}$ is computed from a rank-one approximation of the optimal solution of (3), as explained in part (b) above.

(iv) $\mathbf{x}^{(\mathrm{d})}$ is computed by rounding 100 samples of $\mathcal{N}\big(\mathbf{z}, \mathbf{Z} - \mathbf{z}\mathbf{z}^T\big)$, as explained in part (c) above.

Generate a table listing the values of the least-squares metric

$$f(\mathbf{x}) = ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2 \ ,$$

for each of the heuristic solutions $\mathbf{x}^{(\mathrm{a})}$, $\mathbf{x}^{(\mathrm{b})}$, $\mathbf{x}^{(\mathrm{c})}$, and $\mathbf{x}^{(\mathrm{d})}$, as well as the true input $\widehat{\mathbf{x}}$. Include in this table the lower bound provided by the SDP relaxation given in (3).

Hint: To generate samples of $\mathcal{N}\big(\mathbf{z}, \mathbf{Z} - \mathbf{z}\mathbf{z}^T\big)$, consider using the following result from probability theory. If $\mathbf{u} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then the affine transformation $\mathbf{w} \triangleq \mathbf{B}\mathbf{u} + \mathbf{c}$ is such that $\mathbf{w} \sim \mathcal{N}\big(\mathbf{B}\boldsymbol{\mu} + \mathbf{c}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^T\big)$. Thus, if $\mathbf{u} \sim \mathcal{N}(\mathbf{0}_{n\times 1}, \mathbf{I}_n)$, then $\mathbf{w} \triangleq (\mathbf{Z} - \mathbf{z}\mathbf{z})^{1/2}\,\mathbf{u} + \mathbf{z}$ is such that $\mathbf{w} \sim \mathcal{N}\big(\mathbf{z}, \mathbf{Z} - \mathbf{z}\mathbf{z}^T\big)$.

Reading assignments:

1. Read through Chapters 7 and 8 of CO-BV. Look over parts of CVX as necessary.

Reminders:

Late homework policy for EE 150: Late homeworks will not be accepted. There will be no exceptions to this other than institute established emergency reasons, in which case a signed letter is required from an authorized official.

NCT Problems: Remember that problems with an asterisk, such as ***7** are no collaboration type (NCT) problems.

Texts: The abbreviation CO-BV corresponds to the textbook "*Convex Optimization*" by Stephen Boyd and Lieven Vandenberghe. In addition, CO-AE refers to the *Additional Exercises for Convex Optimization*, also by Boyd and Vandenberghe. Finally, CVX corresponds to the `cvx` *Users' Guide* by Michael Grant and Stephen Boyd.
