EE 150 - Applications of Convex Optimization in Signal Processing and Communications
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Homework Set #4 - Solutions

1. (a) We have

$$\log(f(x)) = \log\left(\frac{e^x}{1+e^x}\right) = x - \log(1+e^x) \ .$$

The first term is linear and hence concave. Since the function $\log(1+e^x)$ is strictly convex, as it is the log-sum-exp function evaluated at $x_1 = 0$, $x_2 = x$, the second term above is strictly concave. Hence, $f(x) = \frac{e^x}{1+e^x}$ is strictly log-concave.

(b) Note that the first and second partial derivatives of

$$h(\mathbf{x}) \triangleq \log(f(\mathbf{x})) = -\log\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)$$

are given by the following:

$$\begin{aligned} \frac{\partial h(\mathbf{x})}{\partial x_k} &= \frac{\frac{1}{x_k^2}}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} \ , \\ \frac{\partial^2 h(\mathbf{x})}{\partial x_k \partial x_\ell} &= \begin{cases} \dfrac{-\frac{2}{x_k^3}}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} + \dfrac{\frac{1}{x_k^4}}{\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)^2} \ , & k = \ell \\ \dfrac{\frac{1}{x_k^2 x_\ell^2}}{\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)^2} \ , & k \neq \ell \end{cases} . \end{aligned}$$

To show that $f(\mathbf{x})$ is log-concave, we must show that $h(\mathbf{x})$ is concave, which is equivalent to proving that the Hessian of $h(\mathbf{x})$ is negative semidefinite, i.e., $\nabla^2 h(\mathbf{x}) \preceq \mathbf{0}$. This is equivalent to proving that $\mathbf{y}^T \left(\nabla^2 h(\mathbf{x})\right) \mathbf{y} \leq 0$ for all $\mathbf{y}$, which in turn is equivalent to proving

$$\left(\sum_{\ell=1}^{n} \frac{1}{x_\ell}\right)\left(\sum_{k=1}^{n} \left(-\frac{2y_k^2}{x_k^3}\right)\right) + \left(\sum_{k=1}^{n} \frac{y_k}{x_k^2}\right)\left(\sum_{\ell=1}^{n} \frac{y_\ell}{x_\ell^2}\right) \leq 0 \, ,$$

in light of the domain of $f(\mathbf{x})$. The above condition can be simplified to the following:

$$\left(\sum_{k=1}^{n} \frac{y_k}{x_k^2}\right)^2 \leq 2\left(\sum_{k=1}^{n} \frac{1}{x_k}\right)\left(\sum_{k=1}^{n} \frac{y_k^2}{x_k^3}\right) \ . \tag{1}$$

But recall that for the standard inner product over $\mathbb{R}^n$, we have $\left(\mathbf{a}^T\mathbf{b}\right)^2 \leq ||\mathbf{a}||_2^2 \, ||\mathbf{b}||_2^2$ by the Cauchy-Schwarz inequality. By choosing

$$a_k = \frac{1}{\sqrt{x_k}} \ , \ b_k = \frac{y_k}{x_k\sqrt{x_k}} \ ,$$

we get

$$\left(\sum_{k=1}^{n} \frac{y_k}{x_k^2}\right)^2 \leq \left(\sum_{k=1}^{n} \frac{1}{x_k}\right)\left(\sum_{k=1}^{n} \frac{y_k^2}{x_k^3}\right) \ , \tag{2}$$

with equality if and only if $y_k = x_k$. On account of the domain of $f(\mathbf{x})$ and (2), it follows that for $\mathbf{y} \neq 0$, we have the strict inequality

$$\sum_{k=1}^{n} \left(\frac{y_k}{x_k^2}\right)^2 < 2 \left(\sum_{k=1}^{n} \frac{1}{x_k}\right) \left(\sum_{k=1}^{n} \frac{y_k^2}{x_k^3}\right) .$$

When this is compared to the condition given in (1), this proves that $\mathbf{y}^T \left(\nabla^2 h(\mathbf{x})\right) \mathbf{y} < 0$ for all $\mathbf{y} \neq \mathbf{0}$, which means that $h(\mathbf{x}) = \log(f(\mathbf{x}))$ is *strictly* concave over its domain. Thus, the function $f(\mathbf{x})$ is *strictly* log-concave over its domain.

(c) We show that the function

$$g(\mathbf{x}) \triangleq \log(f(\mathbf{x})) = \sum_{k=1}^{n} \log(x_k) - \log\left(\sum_{k=1}^{n} x_k\right)$$

is strictly concave over the domain of $f(\mathbf{x})$. Taking partial derivatives yields the following:

$$\begin{aligned} \frac{\partial g(\mathbf{x})}{\partial x_k} &= \frac{1}{x_k} - \frac{1}{(\mathbf{1}^T\mathbf{x})} , \\ \frac{\partial^2 g(\mathbf{x})}{\partial x_k \partial x_\ell} &= \begin{cases} -\dfrac{1}{x_k^2} + \dfrac{1}{(\mathbf{1}^T\mathbf{x})^2} , & k = \ell \\ \dfrac{1}{(\mathbf{1}^T\mathbf{x})^2} , & k \neq \ell \end{cases} \end{aligned}$$

From this, the Hessian of $g(\mathbf{x})$ can be expressed as follows:

$$\nabla^2 g(\mathbf{x}) = -\left(\mathrm{diag}(\mathbf{x})\right)^{-2} + \frac{1}{(\mathbf{1}^T\mathbf{x})^2}\mathbf{1}\mathbf{1}^T = \left(\mathrm{diag}(\mathbf{x})\right)^{-1}\left(-\mathbf{I} + \frac{1}{(\mathbf{1}^T\mathbf{x})^2}\mathbf{x}\mathbf{x}^T\right)\left(\mathrm{diag}(\mathbf{x})\right)^{-1}$$

In light of the domain of $f(\mathbf{x})$, to prove that $\nabla^2 g(\mathbf{x}) \preceq \mathbf{0}$, it is equivalent to show that $\left(-\mathbf{I} + \mathbf{w}\mathbf{w}^T\right) \preceq \mathbf{0}$, where $\mathbf{w} \triangleq \frac{1}{(\mathbf{1}^T\mathbf{x})}\mathbf{x}$. Equivalently, we need to show that

$$\mathbf{v}^T \left(-\mathbf{I} + \mathbf{w}\mathbf{w}^T\right) \mathbf{v} \leq 0 \Longleftrightarrow \left(\mathbf{v}^T\mathbf{w}\right)^2 \leq ||\mathbf{v}||_2^2 , \tag{3}$$

for any $\mathbf{v}$. Prior to proving (3), we will show that $||\mathbf{w}||_2 < ||\mathbf{w}||_1 = 1$ here. Note that as $\mathbf{w} = \frac{1}{(\mathbf{1}^T\mathbf{x})}\mathbf{x}$, we have $\mathbf{w} \succ \mathbf{0}$ on account of the domain of $f(\mathbf{x})$ and $||\mathbf{w}||_1 = \mathbf{1}^T\mathbf{w} = \frac{(\mathbf{1}^T\mathbf{x})}{(\mathbf{1}^T\mathbf{x})} = 1$. Also, we have

$$1 = ||\mathbf{w}||_1^2 = \left(\sum_{k=1}^{n} w_k\right)^2 = \sum_{k=1}^{n}\sum_{\ell=1}^{n} w_k w_\ell = \underbrace{\sum_{k=1}^{n} w_k^2}_{||\mathbf{w}||_2^2} + \underbrace{\sum_{k=1}^{n}\sum_{\ell\neq k} w_k w_\ell}_{>0 \text{ as } \mathbf{w}\succ\mathbf{0}} .$$

This proves that $||\mathbf{w}||_2 < 1$ here. Returning to (3), we will assume that $\mathbf{v} \neq 0$, since the condition trivially holds for $\mathbf{v} \neq 0$. Recall that from the Cauchy-Schwarz inequality, we have

$$\left(\mathbf{v}^T\mathbf{w}\right)^2 \leq ||\mathbf{v}||_2^2 \, ||\mathbf{w}||_2^2 ,$$

for any $\mathbf{v}, \mathbf{w}$ with equality if and only if $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent. However, as $||\mathbf{w}||_2 < 1$, it follows that $\left(\mathbf{v}^T\mathbf{w}\right)^2 < ||\mathbf{v}||_2^2$ for any $\mathbf{v} \neq \mathbf{0}$. Hence, we have shown that for any $\mathbf{v} \neq \mathbf{0}$, we have

$$\left(\mathbf{v}^T\mathbf{w}\right)^2 < ||\mathbf{v}||_2^2 \Longleftrightarrow \mathbf{v}^T\left(-\mathbf{I} + \mathbf{w}\mathbf{w}^T\right)\mathbf{v} < 0 \Longleftrightarrow \left(-\mathbf{I} + \mathbf{w}\mathbf{w}^T\right) \prec \mathbf{0} \Longleftrightarrow \nabla^2 g(\mathbf{x}) \prec \mathbf{0}\,.$$

Thus, $g(\mathbf{x}) = \log(f(\mathbf{x}))$ is *strictly* concave, which proves that $f(\mathbf{x})$ is strictly log-concave.

(d) We prove that

$$h(\mathbf{X}) \triangleq \log(f(\mathbf{X})) = \log(\det(\mathbf{X})) - \log(\operatorname{tr}(\mathbf{X}))$$

is strictly concave over the domain of $f(\mathbf{X})$. Consider the restriction to a line $\mathbf{X} = \mathbf{Z} + t\mathbf{V}$, where $\mathbf{Z} \succ \mathbf{0}$ and $\mathbf{V} \in \mathbb{S}^n$, and use the eigenvalue decomposition

$$\mathbf{Z}^{-\frac{1}{2}}\mathbf{V}\mathbf{Z}^{-\frac{1}{2}} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T = \sum_{k=1}^{n} \lambda_k \mathbf{q}_k \mathbf{q}_k^T\,.$$

Then, we have

$$\begin{aligned}
h(\mathbf{Z} + t\mathbf{V}) &= \log(\det(\mathbf{Z} + t\mathbf{V})) - \log(\operatorname{tr}(\mathbf{Z} + t\mathbf{V}))\,, \\
&= \log(\det(\mathbf{Z})) + \log\left(\det\left(\mathbf{I} + t\mathbf{Z}^{-\frac{1}{2}}\mathbf{V}\mathbf{Z}^{-\frac{1}{2}}\right)\right) - \log\left(\operatorname{tr}\left(\mathbf{Z}\left(\mathbf{I} + t\mathbf{Z}^{-\frac{1}{2}}\mathbf{V}\mathbf{Z}^{-\frac{1}{2}}\right)\right)\right)\,, \\
&= \log(\det(\mathbf{Z})) + \sum_{k=1}^{n} \log(1 + t\lambda_k) - \log\left(\sum_{k=1}^{n} \left(\mathbf{q}_k^T\mathbf{Z}\mathbf{q}_k\right)(1 + t\lambda_k)\right)\,, \\
&= \log(\det(\mathbf{Z})) - \sum_{k=1}^{n} \log\left(\mathbf{q}_k^T\mathbf{Z}\mathbf{q}_k\right) \\
&\quad + \sum_{k=1}^{n} \log\left(\left(\mathbf{q}_k^T\mathbf{Z}\mathbf{q}_k\right)(1 + t\lambda_k)\right) - \log\left(\sum_{k=1}^{n} \left(\mathbf{q}_k^T\mathbf{Z}\mathbf{q}_k\right)(1 + t\lambda_k)\right)\,.
\end{aligned}$$

From this last expression for $h(\mathbf{Z} + t\mathbf{V})$, note that it is a constant, plus the function

$$g(\mathbf{y}) \triangleq \sum_{k=1}^{n} \log(y_k) - \log\left(\sum_{k=1}^{n} y_k\right)\,,$$

evaluated at $y_k = \left(\mathbf{q}_k^T\mathbf{Z}\mathbf{q}_k\right)(1 + t\lambda_k)$. But we showed in part (c) that $g(\mathbf{y})$ was strictly concave. Hence, it follows that $h(\mathbf{X})$ is strictly concave, which means that $f(\mathbf{X})$ is strictly log-concave.

2. (a) i. The domain of the objective is convex, because $f_0(\mathbf{x})$ is convex. The sublevel sets are convex because $\frac{f_0(\mathbf{x})}{\mathbf{c}^T\mathbf{x}+d} \leq \alpha$ if and only if $\mathbf{c}^T\mathbf{x} + d > 0$ and $f_0(\mathbf{x}) - \alpha\left(\mathbf{c}^T\mathbf{x} + d\right) \leq 0$, which is always a convex inequality for any α. Hence, the problem is a quasiconvex optimization problem.

ii. Suppose $\mathbf{x}$ is feasible in the original problem. Define $t \triangleq \frac{1}{\mathbf{c}^T\mathbf{x}+d}$ and $\mathbf{y} \triangleq \frac{\mathbf{x}}{\mathbf{c}^T\mathbf{x}+d}$. Then $t > 0$ and it is easily verified that t and $\mathbf{y}$ are feasible in the transformed problem, with the objective value $g_0(\mathbf{y}, t) = \frac{f_0(\mathbf{x})}{\mathbf{c}^T\mathbf{x}+d}$.
Conversely, suppose $\mathbf{y}$ and t are feasible for the transformed problem. We must have $t > 0$, by definition of the domain of the perspective function. Define

$\mathbf{x} \triangleq \mathbf{y}/t$. We have $\mathbf{x} \in \mathrm{dom}(f_k(\mathbf{x}))$ for $k = 0, 1, \ldots, m$ (again, by definition of the perspective function). Note that $\mathbf{x}$ is feasible in the original problem, because

$$f_k(\mathbf{x}) = g_k(\mathbf{y}, t)\,/t \leq 0\,,\ \ k = 1, \ldots, m\,,\ \ \mathbf{Ax} = \mathbf{A}\,(\mathbf{y}/t) = \mathbf{b}\,.$$

From the last equality, $\mathbf{c}^T\mathbf{x} + d = \left(\mathbf{c}^T\mathbf{y} + td\right)/t = 1/t$, and hence,

$$t = \frac{1}{\mathbf{c}^T\mathbf{x} + d}\,,\ \ \frac{f_0(\mathbf{x})}{\mathbf{c}^T\mathbf{x} + d} = t f_0(\mathbf{x}) = g_0(\mathbf{y}, t)\ .$$

Therefore, $\mathbf{x}$ is feasible in the original problem, with the objective value $g_0(\mathbf{y}, t)$. In conclusion, from any feasible point of one problem, we can derive a feasible point of the other problem with the same objective value.
Clearly the transformed problem is convex as the domain is a convex set, the objective and inequality constraint functions are convex, and the equality constraints are affine.

(b) i. The domain of the objective function is convex, and its sublevel sets are convex because for $\alpha \geq 0$, we have $\frac{f_0(\mathbf{x})}{h(\mathbf{x})} \leq \alpha$ if and only if $f_0(\mathbf{x}) - \alpha h(\mathbf{x}) \leq 0$, which is a convex inequality. For $\alpha < 0$, the sublevel sets are empty. Combining both results, we find that the problem is a quasiconvex optimization problem.

ii. To verify the equivalence, assume first that $\mathbf{x}$ is feasible in the original problem. Define $t \triangleq \frac{1}{h(\mathbf{x})}$ and $\mathbf{y} \triangleq \frac{\mathbf{x}}{h(\mathbf{x})}$. Then, $t > 0$ and

$$g_k(\mathbf{y}, t) = t f_k(\mathbf{y}/t) = t f_k(\mathbf{x}) \leq 0\,,\ \ k = 1, \ldots, m\,,\ \ \mathbf{Ay} = \mathbf{A}\left(\frac{\mathbf{x}}{h(\mathbf{x})}\right) = \mathbf{b}t\,.$$

Moreover, $\widetilde{h}(\mathbf{y}, t) = t h(\mathbf{y}/t) = \frac{h(\mathbf{x})}{h(\mathbf{x})} = 1$ and $g_0(\mathbf{y}, t) = t f_0(\mathbf{y}/t) = \frac{f_0(\mathbf{x})}{h(\mathbf{x})}$. We see that for every feasible point in the original problem, we can find a feasible point in the transformed problem with the same objective value.
Conversely, assume $\mathbf{y}$ and t are feasible in the transformed problem. By definition of the perspective function, $t > 0$. Define $\mathbf{x} \triangleq \mathbf{y}/t$. We have

$$f_k(\mathbf{x}) = f_k(\mathbf{y}/t) = g_k(\mathbf{y}, t)\,/t \leq 0\,,\ \ k = 1, \ldots, m\,,\ \ \mathbf{Ax} = \mathbf{A}\,(\mathbf{y}/t) = \mathbf{b}\,.$$

From the last inequality, we have

$$\widetilde{h}(\mathbf{y}, t) = -t h(\mathbf{y}/t) = -t h(\mathbf{x}) \leq -1\,.$$

This implies that $h(\mathbf{x}) > 0$ and $t h(\mathbf{x}) \geq 1$. Finally, the objective is

$$\frac{f_0(\mathbf{x})}{h(\mathbf{x})} = \frac{g_0(\mathbf{y}, t)}{t h(\mathbf{x})} \leq g_0(\mathbf{y}, t)\ .$$

We conclude that with every feasible point in the transformed problem, there is a corresponding feasible point in the original problem with the same or lower objective value.
Putting the two parts together, we can conclude that the two problems have the same optimal value, and that optimal solutions for one problem are optimal for the other (if both are solvable).
Clearly the transformed problem is convex as the domain is a convex set, the objective and inequality constraint functions are convex, and the equality constraints are affine.

iii. Exploiting the fact that $\alpha \mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\alpha\mathbf{A})$ and $\det(\alpha\mathbf{A}) = \alpha^m \det(\mathbf{A})$ for any scalar α and $m \times m$ matrix $\mathbf{A}$, we get the following after some algebraic manipulations:

$$\begin{array}{ll} \text{minimize} & \dfrac{\frac{1}{m}\mathrm{tr}(t\mathbf{F}_0 + y_1\mathbf{F}_1 + \cdots + y_n\mathbf{F}_n)}{\left(\det(t\mathbf{F}_0 + y_1\mathbf{F}_1 + \cdots + y_n\mathbf{F}_n)\right)^{\frac{1}{m}}} \end{array} ,$$

with domain

$$\{(\mathbf{y}, t) : t > 0\,,\ t\mathbf{F}_0 + y_1\mathbf{F}_1 + \cdots + y_n\mathbf{F}_n \succ \mathbf{0}\} \,.$$

3. – (*Complex ℓ_1-norm:*)
For simplicity, let us express $\mathbf{A}$ in terms of its rows as

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix} ,$$

where $\mathbf{a}_k \in \mathbb{C}^n$ for $k = 1, \ldots, m$. Then, note that the problem

$$\text{minimize} \quad ||\mathbf{A}\mathbf{x} - \mathbf{b}||_1 = \sum_{k=1}^{m} \left|\mathbf{a}_k^T\mathbf{x} - b_k\right| \,,$$

is equivalent to the following one:

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{m} t_k \\ \text{subject to} & \left|\mathbf{a}_k^T\mathbf{x} - b_k\right| \le t_k\,,\ k = 1, \ldots, m \end{array} . \tag{4}$$

Here, we have introduced new optimization variables $t_1, \ldots, t_m$. To simplify $\left|\mathbf{a}_k^T\mathbf{x} - b_k\right|$, note that we have

$$\left|\mathbf{a}_k^T\mathbf{x} - b_k\right|^2 = \left(\mathrm{Re}\left[\mathbf{a}_k^T\mathbf{x} - b_k\right]\right)^2 + \left(\mathrm{Im}\left[\mathbf{a}_k^T\mathbf{x} - b_k\right]\right)^2 .$$

But we have

$$\begin{aligned} \mathbf{a}_k^T\mathbf{x} - b_k &= \left(\mathrm{Re}\left[\mathbf{a}_k^T\right] + j\mathrm{Im}\left[\mathbf{a}_k^T\right]\right)\left(\mathrm{Re}[\mathbf{x}] + j\mathrm{Im}[\mathbf{x}]\right) - \left(\mathrm{Re}[b_k] + j\mathrm{Im}[b_k]\right) , \\ &= \underbrace{\left(\mathrm{Re}\left[\mathbf{a}_k^T\right]\mathrm{Re}[\mathbf{x}] - \mathrm{Im}\left[\mathbf{a}_k^T\right]\mathrm{Im}[\mathbf{x}] - \mathrm{Re}[b_k]\right)}_{\mathrm{Re}\left[\mathbf{a}_k^T\mathbf{x} - b_k\right]} \\ &\quad + j\underbrace{\left(\mathrm{Im}\left[\mathbf{a}_k^T\right]\mathrm{Re}[\mathbf{x}] + \mathrm{Re}\left[\mathbf{a}_k^T\right]\mathrm{Im}[\mathbf{x}] - \mathrm{Im}[b_k]\right)}_{\mathrm{Im}\left[\mathbf{a}_k^T\mathbf{x} - b_k\right]} . \end{aligned}$$

Substituting this into the expression above yields

$$\begin{aligned} \left|\mathbf{a}_k^T\mathbf{x} - b_k\right|^2 &= \left(\left(\mathrm{Re}\left[\mathbf{a}_k^T\right]\mathrm{Re}[\mathbf{x}] - \mathrm{Im}\left[\mathbf{a}_k^T\right]\mathrm{Im}[\mathbf{x}] - \mathrm{Re}[b_k]\right)\right)^2 \\ &\quad + \left(\left(\mathrm{Im}\left[\mathbf{a}_k^T\right]\mathrm{Re}[\mathbf{x}] + \mathrm{Re}\left[\mathbf{a}_k^T\right]\mathrm{Im}[\mathbf{x}] - \mathrm{Im}[b_k]\right)\right)^2 \end{aligned} \tag{5}$$

If we define the following real quantities:

$$\widetilde{\mathbf{A}}_k \triangleq \begin{bmatrix} \mathrm{Re}\left[\mathbf{a}_k^T\right] & -\mathrm{Im}\left[\mathbf{a}_k^T\right] \\ \mathrm{Im}\left[\mathbf{a}_k^T\right] & \mathrm{Re}\left[\mathbf{a}_k^T\right] \end{bmatrix} ,\ \widetilde{\mathbf{x}} \triangleq \begin{bmatrix} \mathrm{Re}[\mathbf{x}] \\ \mathrm{Im}[\mathbf{x}] \end{bmatrix} ,\ \widetilde{\mathbf{b}}_k \triangleq \begin{bmatrix} \mathrm{Re}[b_k] \\ \mathrm{Im}[b_k] \end{bmatrix} ,\ k = 1, \ldots, m ,$$

then from (5), it can be shown that we have

$$\left|\mathbf{a}_k^T\mathbf{x} - b_k\right|^2 = \left|\left|\widetilde{\mathbf{A}}_k\widetilde{\mathbf{x}} - \widetilde{\mathbf{b}}_k\right|\right|_2^2 \Longleftrightarrow \left|\mathbf{a}_k^T\mathbf{x} - b_k\right| = \left|\left|\widetilde{\mathbf{A}}_k\widetilde{\mathbf{x}} - \widetilde{\mathbf{b}}_k\right|\right|_2 .$$

Hence, from (4), the complex ℓ_1-norm approximation problem can be equivalently expressed as

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^{m} t_k \\ \text{subject to} & \left|\left|\widetilde{\mathbf{A}}_k\widetilde{\mathbf{x}} - \widetilde{\mathbf{b}}_k\right|\right|_2 \leq t_k\,,\ k = 1, \ldots, m \end{array} ,$$

where $\widetilde{\mathbf{A}}_k \in \mathbb{R}^{2\times 2n}$ and $\widetilde{\mathbf{b}}_k \in \mathbb{R}^2$ for $k = 1, \ldots, m$ are the real problem data and $t_k \in \mathbb{R}$ for $k = 1, \ldots, m$ and $\widetilde{\mathbf{x}} \in \mathbb{R}^{2n}$ are the real variables. But this is just an SOCP.

– (*Complex ℓ_2-norm:*)
Minimizing $||\mathbf{Ax} - \mathbf{b}||_2$ is equivalent to minimizing its square. So, let us expand $||\mathbf{Ax} - \mathbf{b}||_2^2$ around the real and imaginary parts of $\mathbf{Ax} - \mathbf{b}$. We get

$$||\mathbf{Ax} - \mathbf{b}||_2^2 = ||\text{Re}[\mathbf{Ax} - b]||_2^2 + ||\text{Im}[\mathbf{Ax} - b]||_2^2 .$$

But we have

$$\begin{aligned} \mathbf{Ax} - \mathbf{b} &= (\text{Re}[\mathbf{A}] + j\text{Im}[\mathbf{A}])\,(\text{Re}[\mathbf{x}] + j\text{Im}[\mathbf{x}]) - (\text{Re}[\mathbf{b}] + j\text{Im}[\mathbf{b}])\ , \\ &= \underbrace{(\text{Re}[\mathbf{A}]\,\text{Re}[\mathbf{x}] - \text{Im}[\mathbf{A}]\,\text{Im}[\mathbf{x}] - \text{Re}[\mathbf{b}])}_{\text{Re}[\mathbf{Ax}-\mathbf{b}]} \\ &\quad + j\,\underbrace{(\text{Im}[\mathbf{A}]\,\text{Re}[\mathbf{x}] + \text{Re}[\mathbf{A}]\,\text{Im}[\mathbf{x}] - \text{Im}[\mathbf{b}])}_{\text{Im}[\mathbf{Ax}-\mathbf{b}]} . \end{aligned}$$

Substituting this into the expression above yields

$$\begin{aligned} ||\mathbf{Ax} - \mathbf{b}||_2^2 &= ||(\text{Re}[\mathbf{A}]\,\text{Re}[\mathbf{x}] - \text{Im}[\mathbf{A}]\,\text{Im}[\mathbf{x}] - \text{Re}[\mathbf{b}])||_2^2 \\ &\quad + ||(\text{Im}[\mathbf{A}]\,\text{Re}[\mathbf{x}] + \text{Re}[\mathbf{A}]\,\text{Im}[\mathbf{x}] - \text{Im}[\mathbf{b}])||_2^2 . \end{aligned} \qquad (6)$$

If we define the following real quantities:

$$\widetilde{\mathbf{A}} \triangleq \begin{bmatrix} \text{Re}[\mathbf{A}] & -\text{Im}[\mathbf{A}] \\ \text{Im}[\mathbf{A}] & \text{Re}[\mathbf{A}] \end{bmatrix},\ \widetilde{\mathbf{x}} \triangleq \begin{bmatrix} \text{Re}[\mathbf{x}] \\ \text{Im}[\mathbf{x}] \end{bmatrix},\ \widetilde{\mathbf{b}} \triangleq \begin{bmatrix} \text{Re}[\mathbf{b}] \\ \text{Im}[\mathbf{b}] \end{bmatrix},$$

then from (6), it can be shown that we have

$$||\mathbf{Ax} - \mathbf{b}||_2^2 = \left|\left|\widetilde{\mathbf{A}}\widetilde{\mathbf{x}} - \widetilde{\mathbf{b}}\right|\right|_2^2 = \widetilde{\mathbf{x}}^T\widetilde{\mathbf{A}}^T\widetilde{\mathbf{A}}\widetilde{\mathbf{x}} - 2\widetilde{\mathbf{b}}^T\widetilde{\mathbf{A}}\widetilde{\mathbf{x}} + \widetilde{\mathbf{b}}^T\widetilde{\mathbf{b}}.$$

Hence, the complex ℓ_2-norm approximation problem can be equivalently expressed as

$$\text{minimize}\quad \widetilde{\mathbf{x}}^T\widetilde{\mathbf{A}}^T\widetilde{\mathbf{A}}\widetilde{\mathbf{x}} - 2\widetilde{\mathbf{b}}^T\widetilde{\mathbf{A}}\widetilde{\mathbf{x}} + \widetilde{\mathbf{b}}^T\widetilde{\mathbf{b}}\ ,$$

where $\widetilde{\mathbf{A}} \in \mathbb{R}^{2m\times 2n}$ and $\widetilde{\mathbf{b}} \in \mathbb{R}^{2m}$ are the real problem data and $\widetilde{\mathbf{x}} \in \mathbb{R}^{2n}$ is the real variable. But this is just a QCQP.

– (*Complex ℓ_∞-norm:*)
For simplicity, let us express $\mathbf{A}$ in terms of its rows as

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix},$$

where $\mathbf{a}_k \in \mathbb{C}^n$ for $k = 1, \ldots, m$. Then, note that the problem

$$\text{minimize} \quad ||\mathbf{A}\mathbf{x} - \mathbf{b}||_\infty = \max_{k=1,\ldots,m} \left|\mathbf{a}_k^T \mathbf{x} - b_k\right| \ ,$$

is equivalent to the following one:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left|\mathbf{a}_k^T \mathbf{x} - b_k\right| \leq t\,, \ k = 1, \ldots, m \end{array} . \tag{7}$$

Here, we have introduced a new optimization variable t. To simplify $\left|\mathbf{a}_k^T \mathbf{x} - b_k\right|$, note that we have

$$\left|\mathbf{a}_k^T \mathbf{x} - b_k\right|^2 = \left(\text{Re}\left[\mathbf{a}_k^T \mathbf{x} - b_k\right]\right)^2 + \left(\text{Im}\left[\mathbf{a}_k^T \mathbf{x} - b_k\right]\right)^2 .$$

But we have

$$\begin{aligned} \mathbf{a}_k^T \mathbf{x} - b_k &= \left(\text{Re}\left[\mathbf{a}_k^T\right] + j\text{Im}\left[\mathbf{a}_k^T\right]\right)(\text{Re}[\mathbf{x}] + j\text{Im}[\mathbf{x}]) - (\text{Re}[b_k] + j\text{Im}[b_k]) \ , \\ &= \underbrace{\left(\text{Re}\left[\mathbf{a}_k^T\right]\text{Re}[\mathbf{x}] - \text{Im}\left[\mathbf{a}_k^T\right]\text{Im}[\mathbf{x}] - \text{Re}[b_k]\right)}_{\text{Re}\left[\mathbf{a}_k^T \mathbf{x} - b_k\right]} \\ &\quad + j\underbrace{\left(\text{Im}\left[\mathbf{a}_k^T\right]\text{Re}[\mathbf{x}] + \text{Re}\left[\mathbf{a}_k^T\right]\text{Im}[\mathbf{x}] - \text{Im}[b_k]\right)}_{\text{Im}\left[\mathbf{a}_k^T \mathbf{x} - b_k\right]} . \end{aligned}$$

Substituting this into the expression above yields

$$\begin{aligned} \left|\mathbf{a}_k^T \mathbf{x} - b_k\right|^2 &= \left(\left(\text{Re}\left[\mathbf{a}_k^T\right]\text{Re}[\mathbf{x}] - \text{Im}\left[\mathbf{a}_k^T\right]\text{Im}[\mathbf{x}] - \text{Re}[b_k]\right)\right)^2 \\ &\quad + \left(\left(\text{Im}\left[\mathbf{a}_k^T\right]\text{Re}[\mathbf{x}] + \text{Re}\left[\mathbf{a}_k^T\right]\text{Im}[\mathbf{x}] - \text{Im}[b_k]\right)\right)^2 \end{aligned} \tag{8}$$

If we define the following real quantities:

$$\widetilde{\mathbf{A}}_k \triangleq \begin{bmatrix} \text{Re}\left[\mathbf{a}_k^T\right] & -\text{Im}\left[\mathbf{a}_k^T\right] \\ \text{Im}\left[\mathbf{a}_k^T\right] & \text{Re}\left[\mathbf{a}_k^T\right] \end{bmatrix}, \ \widetilde{\mathbf{x}} \triangleq \begin{bmatrix} \text{Re}[\mathbf{x}] \\ \text{Im}[\mathbf{x}] \end{bmatrix}, \ \widetilde{\mathbf{b}}_k \triangleq \begin{bmatrix} \text{Re}[b_k] \\ \text{Im}[b_k] \end{bmatrix}, \ k = 1, \ldots, m\,,$$

then from (8), it can be shown that we have

$$\left|\mathbf{a}_k^T \mathbf{x} - b_k\right|^2 = \left|\left|\widetilde{\mathbf{A}}_k \widetilde{\mathbf{x}} - \widetilde{\mathbf{b}}_k\right|\right|_2^2 \Longleftrightarrow \left|\mathbf{a}_k^T \mathbf{x} - b_k\right| = \left|\left|\widetilde{\mathbf{A}}_k \widetilde{\mathbf{x}} - \widetilde{\mathbf{b}}_k\right|\right|_2 .$$

Hence, from (7), the complex ℓ_∞-norm approximation problem can be equivalently expressed as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left|\left|\widetilde{\mathbf{A}}_k \widetilde{\mathbf{x}} - \widetilde{\mathbf{b}}_k\right|\right|_2 \leq t\,, \ k = 1, \ldots, m \end{array} ,$$

where $\widetilde{\mathbf{A}}_k \in \mathbb{R}^{2\times 2n}$ and $\widetilde{\mathbf{b}}_k \in \mathbb{R}^2$ for $k = 1, \ldots, m$ are the real problem data and $t \in \mathbb{R}$ and $\widetilde{\mathbf{x}} \in \mathbb{R}^{2n}$ are the real variables. But this is just an SOCP. By squaring both sides of each constraint and defining $\lambda \triangleq t^2$, this problem can be converted into the following QCQP:

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \left\|\widetilde{\mathbf{A}}_k\widetilde{\mathbf{x}} - \widetilde{\mathbf{b}}_k\right\|_2^2 \leq \lambda\,,\ k = 1, \ldots, m \end{array}.$$

4. (a) Note that the problem can be equivalently expressed as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \mathbf{c}^T\left(\mathbf{F}(\mathbf{x})\right)^{-1}\mathbf{c} \leq t \end{array} \iff \begin{array}{ll} \text{minimize} & t \\ \text{subject to} & t - \mathbf{c}^T\left(\mathbf{F}(\mathbf{x})\right)^{-1}\mathbf{c} \geq 0 \end{array}.$$

As $\mathbf{F}(\mathbf{x}) \succ \mathbf{0}$ in the domain of $f(\mathbf{x})$, we can express the constraint above as an LMI by recognizing the left-hand side as an appropriate Schur complement. This yields the following equivalent representation of the problem as an SDP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} \mathbf{F}(\mathbf{x}) & \mathbf{c} \\ \mathbf{c}^T & t \end{bmatrix} \succeq \mathbf{0} \end{array}.$$

(b) Note that the problem can be equivalently expressed as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \mathbf{c}_k^T\left(\mathbf{F}(\mathbf{x})\right)^{-1}\mathbf{c}_k \leq t\,,\ k = 1, \ldots, K \end{array}.$$

Using the same argument from the previous part, we can express this problem as the following SDP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} \mathbf{F}(\mathbf{x}) & \mathbf{c}_k \\ \mathbf{c}_k^T & t \end{bmatrix} \succeq \mathbf{0}\,,\ k = 1, \ldots, K \end{array}.$$

(c) Note that by the Courant-Fischer-Weyl min-max principle, it can be shown that $f(\mathbf{x}) = \lambda_{\max}\left(\left(\mathbf{F}(\mathbf{x})\right)^{-1}\right)$, and so $f(\mathbf{x}) \leq t$ if and only if $\left(\mathbf{F}(\mathbf{x})\right)^{-1} \preceq t\mathbf{I}$, which is equivalent to $t\mathbf{I} - \left(\mathbf{F}(\mathbf{x})\right)^{-1} \succeq \mathbf{0}$. Using a Schur complement, we get the following SDP formulation of the given problem:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} \mathbf{F}(\mathbf{x}) & \mathbf{I} \\ \mathbf{I} & t \end{bmatrix} \succeq \mathbf{0} \end{array}.$$

(d) Let us first simplify the expression for $f(\mathbf{x})$. Note that we have the following:

$$\begin{aligned} f(\mathbf{x}) &= \operatorname{tr}\left(E\left[\mathbf{c}^T\left(\mathbf{F}(\mathbf{x})\right)^{-1}\mathbf{c}\right]\right) = E\left[\operatorname{tr}\left(\mathbf{c}^T\left(\mathbf{F}(\mathbf{x})\right)^{-1}\mathbf{c}\right)\right], \\ &= E\left[\operatorname{tr}\left(\mathbf{c}\mathbf{c}^T\left(\mathbf{F}(\mathbf{x})\right)^{-1}\right)\right] = \operatorname{tr}\left(E\left[\mathbf{c}\mathbf{c}^T\right]\left(\mathbf{F}(\mathbf{x})\right)^{-1}\right). \end{aligned} \tag{9}$$

To simplify $E\big[\mathbf{c}\mathbf{c}^T\big]$, note that we have:

$$\begin{aligned}\mathbf{S} &= E\Big[(\mathbf{c}-\overline{\mathbf{c}})(\mathbf{c}-\overline{\mathbf{c}})^T\Big] = E\big[\mathbf{c}\mathbf{c}^T - \mathbf{c}\overline{\mathbf{c}}^T - \overline{\mathbf{c}}\mathbf{c}^T + \overline{\mathbf{c}}\overline{\mathbf{c}}^T\big] \ , \\ &= E\big[\mathbf{c}\mathbf{c}^T\big] - (E[\mathbf{c}])\,\overline{\mathbf{c}}^T - \overline{\mathbf{c}}\,(E[\mathbf{c}])^T + \overline{\mathbf{c}}\overline{\mathbf{c}}^T \ , \\ &= E\big[\mathbf{c}\mathbf{c}^T\big] - \overline{\mathbf{c}}\overline{\mathbf{c}}^T - \overline{\mathbf{c}}\overline{\mathbf{c}}^T + \overline{\mathbf{c}}\overline{\mathbf{c}}^T = E\big[\mathbf{c}\mathbf{c}^T\big] - \overline{\mathbf{c}}\overline{\mathbf{c}}^T \ .\end{aligned}$$

Hence, $E\big[\mathbf{c}\mathbf{c}^T\big] = \mathbf{S} + \overline{\mathbf{c}}\overline{\mathbf{c}}^T$. Substituting this into (9) yields the following:

$$f(\mathbf{x}) = \overline{\mathbf{c}}^T\left(\mathbf{F}(\mathbf{x})\right)^{-1}\overline{\mathbf{c}} + \mathrm{tr}\Big(\mathbf{S}\left(\mathbf{F}(\mathbf{x})\right)^{-1}\Big) = \overline{\mathbf{c}}^T\left(\mathbf{F}(\mathbf{x})\right)^{-1}\overline{\mathbf{c}} + \sum_{k=1}^{m}\mathbf{s}_k^T\left(\mathbf{F}(\mathbf{x})\right)^{-1}\mathbf{s}_k \ .$$

Hence, the problem of minimizing $f(\mathbf{x})$ can be equivalently expressed as follows:

$$\begin{array}{ll}\text{minimize} & t_0 + \displaystyle\sum_{k=1}^{m} t_k \\ \text{subject to} & \overline{\mathbf{c}}^T\left(\mathbf{F}(\mathbf{x})\right)^{-1}\overline{\mathbf{c}} \le t_0 \\ & \mathbf{s}_k^T\left(\mathbf{F}(\mathbf{x})\right)^{-1}\mathbf{s}_k \le t_k\ ,\ \ k = 1,\ldots,m\end{array} \ .$$

Using appropriate Schur complements, this can be recast as the following SDP:

$$\begin{array}{ll}\text{minimize} & t_0 + \displaystyle\sum_{k=1}^{m} t_k \\ \text{subject to} & \begin{bmatrix}\mathbf{F}(\mathbf{x}) & \overline{\mathbf{c}} \\ \overline{\mathbf{c}}^T & t_0\end{bmatrix} \succeq \mathbf{0} \\ & \begin{bmatrix}\mathbf{F}(\mathbf{x}) & \mathbf{s}_k \\ \mathbf{s}_k^T & t_k\end{bmatrix} \succeq \mathbf{0}\ ,\ \ k = 1,\ldots,m\end{array} \ .$$

***5.** (a) Note that we have $\mathbf{x} = \mathbf{P}\mathbf{y}$ and so

$$x_k = \sum_{\ell=1}^{n} P_{k,\ell}\,y_\ell\ ,\ \text{ where } P_{k,\ell} \ge 0\ ,\ \text{ and } \sum_{\ell=1}^{n} P_{k,\ell} = 1\ ,\ \ k = 1,\ldots,n\ .$$

Hence, by Jensen's inequality, we have

$$f(x_k) = f\left(\sum_{\ell=1}^{n} P_{k,\ell}\,y_\ell\right) \le \sum_{\ell=1}^{n} P_{k,\ell} f(y_\ell)\ ,\ \ k = 1,\ldots,n\ .$$

Summing the above inequality over k yields the following:

$$\sum_{k=1}^{n} f(x_k) \le \sum_{k=1}^{n}\sum_{\ell=1}^{n} P_{k,\ell} f(y_\ell) = \sum_{\ell=1}^{n} f(y_\ell)\sum_{k=1}^{n} P_{k,\ell} = \sum_{\ell=1}^{n} f(y_\ell)\ .$$

Here, we used the fact that $\sum_{k=1}^{n} P_{k,\ell} = 1$ for all ℓ. Thus, we have

$$\sum_{k=1}^{n} f(y_k) \ge \sum_{k=1}^{n} f(x_k)\ ,$$

as desired. Using a more compact matrix notation, if we define the vector function $\mathbf{f}(\mathbf{z}) \triangleq \begin{bmatrix} f(z_1) & \cdots & f(z_n) \end{bmatrix}^T$, then we have

$$\mathbf{1}^T\mathbf{f}(\mathbf{x}) = \mathbf{1}^T\mathbf{f}(\mathbf{P}\mathbf{y}) \leq \mathbf{1}^T\mathbf{P}\mathbf{f}(\mathbf{y}) = \mathbf{1}^T\mathbf{f}(\mathbf{y}) \ .$$

This yields $\mathbf{1}^T\mathbf{f}(\mathbf{y}) \geq \mathbf{1}^T\mathbf{f}(\mathbf{x})$, which was the same result we intended to prove.

(b) Consider the optimization problem

$$\begin{array}{ll} \text{maximize} & \operatorname{tr}(\mathbf{X}\mathbf{Y}) \\ \text{subject to} & \operatorname{tr}(\mathbf{Y}) = r \\ & \mathbf{0} \preceq \mathbf{Y} \preceq \mathbf{I} \end{array} , \qquad (10)$$

with variable $\mathbf{Y} \in \mathbb{S}^n$. Let $\mathbf{X} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$ be the eigenvalue decomposition of $\mathbf{X}$ and make a change of variables $\widehat{\mathbf{Y}} = \mathbf{Q}^T\mathbf{Y}\mathbf{Q}$. Then, we have

$$\operatorname{tr}(\mathbf{X}\mathbf{Y}) = \operatorname{tr}\big(\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T\mathbf{Y}\big) = \operatorname{tr}\Big(\mathbf{\Lambda}\widehat{\mathbf{Y}}\Big) = \sum_{k=1}^{n} \lambda_k \widehat{Y}_{k,k} \ .$$

Also, $\operatorname{tr}(widehat\mathbf{Y}) = \operatorname{tr}(\mathbf{Y})$ and $\mathbf{0} \preceq \mathbf{Y} \preceq \mathbf{I}$ if and only if $\mathbf{0} \preceq \widehat{\mathbf{Y}} \preceq \mathbf{I}$. Thus, the problem reduces to the following equivalent one:

$$\begin{array}{ll} \text{maximize} & \sum_{k=1}^{n} \lambda_k \widehat{Y}_{k,k} \\ \text{subject to} & \sum_{k=1}^{n} \widehat{Y}_{k,k} = r \\ & \mathbf{0} \preceq \widehat{\mathbf{Y}} \preceq \mathbf{I} \end{array} .$$

Next, we note that we can restrict $\widehat{\mathbf{Y}}$ to being diagonal without loss of generality. The reason for this is that if $\widehat{\mathbf{Y}}$ is feasible, then the diagonal matrix with diagonal elements $\widehat{Y}_{k,k}$, namely $\operatorname{diag}\Big(\operatorname{diag}\Big(\widehat{\mathbf{Y}}\Big)\Big)$, is also feasible, and has the same cost function value as $\widehat{\mathbf{Y}}$. The problem then simplifies to

$$\begin{array}{ll} \text{maximize} & \sum_{k=1}^{n} \lambda_k \widehat{Y}_{k,k} \\ \text{subject to} & \sum_{k=1}^{n} \widehat{Y}_{k,k} = r \\ & 0 \leq \widehat{Y}_{k,k} \leq 1 \, , \ k = 1, \ldots, n \end{array} .$$

The inequality constraints limit the diagonal elements of $\widehat{\mathbf{Y}}$ to the interval $[0, 1]$. So, intuitively, to get the largest contribution from the objective function, the largest eigenvalue λ_1 must have a weight of one, i.e., $\widehat{Y}_{1,1} = 1$, the second eigenvalue λ_2 must also have a weight of one (if possible), i.e., $\widehat{Y}_{2,2} = 1$, and so forth until the constraint $\sum_{k=1}^{n} \widehat{Y}_{k,k} = r$ is met. Hence, the optimizating $\widehat{Y}_{k,k}$ are given by

$$\widehat{Y}^\star_{k,k} = \begin{cases} 1 \, , & k = 1, \ldots, r \\ 0 \, , & k = r+1, \ldots, n \end{cases} .$$

From this, we conclude that the solution to the problem above, which is also the solution to (10), is simply

$$\sum_{k=1}^{r} \lambda_k \, .$$

Thus, we have

$$g_r(\mathbf{X}) = \lambda_1(\mathbf{X}) + \cdots + \lambda_r(\mathbf{X}) = \sup\left\{\operatorname{tr}(\mathbf{X}\mathbf{Z}) : \mathbf{Z} \in \mathbb{S}^n \, , \ \mathbf{0} \preceq \mathbf{Z} \preceq \mathbf{I} \, , \ \operatorname{tr}(\mathbf{Z}) = r\right\} \, .$$

Since $\operatorname{tr}(\mathbf{X}\mathbf{Z})$ is a convex function of $\mathbf{Z}$, the above result shows that the function $g_r(\mathbf{X})$ is convex for any $r \in \{1, \ldots, n\}$.

(c) As we showed in the previous part that the function $g_r(\mathbf{X})$ is convex for any $r \in \{1, \ldots, n\}$, we have

$$g_r(\mathbf{X}) = g_r(\theta\mathbf{U} + (1-\theta)\,\mathbf{V}) \leq \theta g_r(\mathbf{U}) + (1-\theta)\, g_r(\mathbf{V}) \ , \ r = 1, \ldots, n \, .$$

But note that we also have

$$\theta g_r(\mathbf{U}) + (1-\theta)\, g_r(\mathbf{V}) = \sum_{k=1}^{r} a_k \, , \ g_r(\mathbf{X}) = \sum_{k=1}^{r} b_k \, .$$

Hence, we have

$$\sum_{k=1}^{r} a_k \geq \sum_{k=1}^{r} b_k \, , \ r = 1, \ldots, n \, .$$

For the special case $r = n$, we have a stronger result, since in this case $g_n(\mathbf{X}) = \operatorname{tr}(\mathbf{X})$. As $\mathbf{X} = \theta\mathbf{U} + (1-\theta)\,\mathbf{V}$, we have $g_n(\mathbf{X}) = \theta g_n(\mathbf{U}) + (1-\theta)\, g_n(\mathbf{V})$. Thus $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k$. Combining all of these results, we find that we have

$$\begin{array}{rcl} \displaystyle\sum_{k=1}^{r} a_k & \geq & \displaystyle\sum_{k=1}^{r} b_k \, , \ r = 1, \ldots, n-1 \\ \displaystyle\sum_{k=1}^{n} a_k & = & \displaystyle\sum_{k=1}^{n} b_k \end{array} \, ,$$

and so $\mathbf{a}$ majorizes $\mathbf{b}$.

(d) Let $\mathbf{X} = \theta\mathbf{U} + (1-\theta)\,\mathbf{V}$. Then, we have the following chain of equalities and inequalities:

$$\begin{aligned} h(\theta\mathbf{U} + (1-\theta)\,\mathbf{V}) &= \sum_{k=1}^{n} f(\lambda_k(\theta\mathbf{U} + (1-\theta)\,\mathbf{V}))\ , \\ &\leq \sum_{k=1}^{n} f(\theta\lambda_k(\mathbf{U}) + (1-\theta)\,\lambda_k(\mathbf{V}))\ , \qquad (11) \\ &\leq \theta\sum_{k=1}^{n} f(\lambda_k(\mathbf{U})) + (1-\theta)\sum_{k=1}^{n} f(\lambda_k(\mathbf{V}))\ , \qquad (12) \\ &= \theta h(\mathbf{U}) + (1-\theta)\, h(\mathbf{V})\ . \qquad (13) \end{aligned}$$

Here, (11) follows from the fact that

$$\begin{bmatrix} \lambda_1(\theta \mathbf{U} + (1-\theta)\,\mathbf{V}) \\ \vdots \\ \lambda_n(\theta \mathbf{U} + (1-\theta)\,\mathbf{V}) \end{bmatrix} \text{ is majorized by } \theta \begin{bmatrix} \lambda_1(\mathbf{U}) \\ \vdots \\ \lambda_n(\mathbf{U}) \end{bmatrix} + (1-\theta) \begin{bmatrix} \lambda_1(\mathbf{V}) \\ \vdots \\ \lambda_n(\mathbf{V}) \end{bmatrix} ,$$

as proved in part (c), along with the results from part (a). Also, (12) follows from the fact that $f(x)$ is convex. Thus, from (13), we have

$$h(\theta \mathbf{U} + (1-\theta)\,\mathbf{V}) \le \theta h(\mathbf{U}) + (1-\theta)\,h(\mathbf{V}) \ ,$$

and so $h(\mathbf{X})$ is convex.