EE/ACM 150 - Applications of Convex Optimization in Signal Processing and Communications
Lecture 18

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May 31, 2012

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Outline

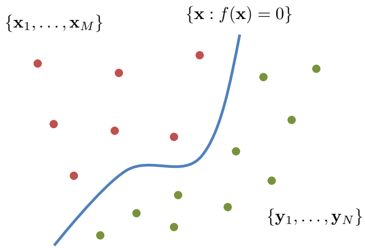
Introduction to Classification

In pattern recognition and classification problems, we are given two sets of points in $\mathbb{R}^n$, say $\{\mathbf{x}_1, \ldots, \mathbf{x}_M\}$ and $\{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$, and we wish to find a function $f : \mathbb{R}^n \to \mathbb{R}$ (say, within a given family of functions), such that we have the following.

$$f(\mathbf{x}_i) > 0\,,\ i = 1, \ldots, M\,,\ f(\mathbf{y}_i) < 0\,,\ i = 1, \ldots, N\,.$$

If these inequalities hold, we say that f (or specifically its 0-level set $\{\mathbf{x} : f(\mathbf{x}) = 0\}$) *separates*, *classifies*, or *discriminates* the two sets of points.

- If only weak versions of the inequalities hold (i.e., $\geq$ and $\leq$, respectively), then we refer to this as *weak separation*.
- This classification problem (including the weak relaxation) need not be feasible.



Linear Discrimination and Interpretations

In *linear discrimination*, we seek an affine function $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} - b$ that classifies the points, i.e.,

$$\mathbf{a}^T\mathbf{x}_i - b > 0\,,\ i = 1, \ldots, M\,,\ \mathbf{a}^T\mathbf{y}_i - b < 0\,,\ i = 1, \ldots, N\,.$$

Geometrically, we seek a hyperplane which separates the points. As the strict inequalities are *homogeneous* in $\mathbf{a}$ and b, they are feasible if and only if the following nonstrict inequalities

$$\mathbf{a}^T\mathbf{x}_i - b \geq 1\,,\ i = 1, \ldots, N\,,\ \mathbf{a}^T\mathbf{y}_i - b \leq -1\,,\ i = 1, \ldots, N\,,$$

are feasible in $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Note that such constraints are affine and, as such, convex.

Linear discrimination alternative interpretation:

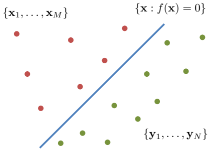
The strong alternative of the set of strict inequalities is the existence of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ such that

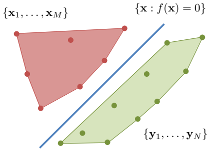
$$\boldsymbol{\lambda} \succeq \mathbf{0}\,,\ \boldsymbol{\mu} \succeq \mathbf{0}\,,\ (\boldsymbol{\lambda}, \boldsymbol{\mu}) \neq \mathbf{0}\,,\ \sum_{i=1}^{M} \lambda_i \mathbf{x}_i = \sum_{i=1}^{N} \mu_i \mathbf{y}_i\,,\ \mathbf{1}^T\boldsymbol{\lambda} = \mathbf{1}^T\boldsymbol{\mu}\,.$$

Here, $\boldsymbol{\lambda} \in \mathbb{R}^M$ and $\boldsymbol{\mu} \in \mathbb{R}^N$. Equivalently, this alternative can be expressed as follows.

$$\boldsymbol{\lambda} \succeq \mathbf{0}\,,\ \mathbf{1}^T\boldsymbol{\lambda} = 1\,,\ \boldsymbol{\mu} \succeq \mathbf{0}\,,\ \mathbf{1}^T\boldsymbol{\mu} = 1\,,\ \sum_{i=1}^{M} \lambda_i \mathbf{x}_i = \sum_{i=1}^{N} \mu_i \mathbf{y}_i\,.$$

This means there is a point in the convex hull of both $\{\mathbf{x}_1, \ldots, \mathbf{x}_M\}$ and $\{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$. So, two sets of points can be linearly discriminated if and only if their convex hulls do not intersect.



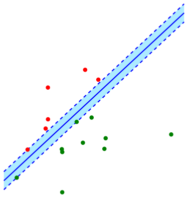


Robust Linear Discrimination

In *robust linear discrimination*, we seek to separate the two sets of points by the maximum possible margin hyperplane (or, equivalently, the thickest slab). As the Euclidean distance between the hyperplanes $\mathcal{H}_1 = \left\{ \mathbf{z} : \mathbf{a}^T \mathbf{z} - b = 1 \right\}$ and $\mathcal{H}_2 = \left\{ \mathbf{z} : \mathbf{a}^T \mathbf{z} - b = -1 \right\}$ is given by $\mathrm{dist}(\mathcal{H}_1, \mathcal{H}_2) = 2 / \left|\left|\mathbf{a}\right|\right|_2$, the maximum margin hyperplane can be found by solving the problem,

$$\begin{array}{ll} \text{minimize} & (1/2) \left|\left|\mathbf{a}\right|\right|_2 \\ \text{subject to} & \mathbf{a}^T \mathbf{x}_i - b \geq 1 \,,\ i = 1, \ldots, M \\ & \mathbf{a}^T \mathbf{y}_i - b \leq -1 \,,\ i = 1, \ldots, N \end{array} ,$$

which (after squaring the objective) becomes a QP in $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$.



Lagrange dual of maximum margin separation problem:

$$\begin{array}{ll} \text{maximize} & \mathbf{1}^T \boldsymbol{\lambda} + \mathbf{1}^T \boldsymbol{\mu} \\ \text{subject to} & 2 \left|\left| \sum_{i=1}^{M} \lambda_i \mathbf{x}_i - \sum_{i=1}^{N} \mu_i \mathbf{y}_i \right|\right|_2 \leq 1 \\ & \mathbf{1}^T \boldsymbol{\lambda} = \mathbf{1}^T \boldsymbol{\mu} \,,\ \boldsymbol{\lambda} \succeq \mathbf{0} \,,\ \boldsymbol{\mu} \succeq \mathbf{0} \end{array} .$$

From duality, the optimal value of the dual is the inverse of the maximum margin of separation.

Interpretation:

- Change variables to $\theta_i = \lambda_i / \mathbf{1}^T \boldsymbol{\lambda}$, $\gamma_i = \mu_i / \mathbf{1}^T \boldsymbol{\mu}$, and $t = 1 / \left(\mathbf{1}^T \boldsymbol{\lambda} + \mathbf{1}^T \boldsymbol{\mu} \right)$.
- Invert the objective to minimize $1 / \left(\mathbf{1}^T \boldsymbol{\lambda} + \mathbf{1}^T \boldsymbol{\mu} \right) = t$, yielding the following problem.

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \left|\left| \sum_{i=1}^{M} \theta_i \mathbf{x}_i - \sum_{i=1}^{N} \gamma_i \mathbf{y}_i \right|\right|_2 \leq t \\ & \boldsymbol{\theta} \succeq \mathbf{0} \,,\ \mathbf{1}^T \boldsymbol{\theta} = 1 \,,\ \boldsymbol{\gamma} \succeq \mathbf{0} \,,\ \mathbf{1}^T \boldsymbol{\gamma} = 1 \end{array} .$$

The optimal value of this problem is the distance between the convex hulls of the sets of points.

Approximate Linear Separation of Non-Separable Sets

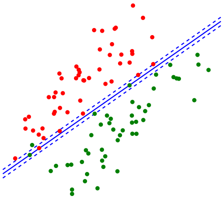
The two sets of points $\{\mathbf{x}_1, \ldots, \mathbf{x}_M\}$ and $\{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$ need not be separable by a hyperplane, which occurs if their convex hulls intersect. In this case, we can relax the classification requirements $\mathbf{a}^T\mathbf{x}_i - b \geq 1$ for $i = 1, \ldots, M$ and $\mathbf{a}^T\mathbf{y}_i - b \leq -1$ for $i = 1, \ldots, N$ by introducing nonnegative variables $\mathbf{u} \in \mathbb{R}^M$ and $\mathbf{v} \in \mathbb{R}^N$ and forming the approximate separation conditions,

$$\mathbf{a}^T\mathbf{x}_i - b \geq 1 - u_i \,, \; i = 1, \ldots, M \,, \; \mathbf{a}^T\mathbf{y}_i - b \leq -\left(1 - v_i\right) \,, \; i = 1, \ldots, N \,.$$

These new conditions can always be made feasible by making $\mathbf{u}$ and $\mathbf{v}$ large enough. Our goal is to find $\mathbf{a}$, b, and *sparse* nonnegative $\mathbf{u}$ and $\mathbf{v}$ that satisfy these inequalities. As a heuristic for this, we can minimize the sum of the components of $\mathbf{u}$ and $\mathbf{v}$, which leads to the following problem.

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T\mathbf{u} + \mathbf{1}^T\mathbf{v} \\ \text{subject to} & \mathbf{a}^T\mathbf{x}_i - b \geq 1 - u_i \,, \; i = 1, \ldots, M \\ & \mathbf{a}^T\mathbf{y}_i - b \leq -\left(1 - v_i\right) \,, \; i = 1, \ldots, N \\ & \mathbf{u} \succeq \mathbf{0} \,, \; \mathbf{v} \succeq \mathbf{0} \end{array} .$$

- This problem is an LP in $\mathbf{a} \in \mathbb{R}^n$, $b \in \mathbb{R}$, $\mathbf{u} \in \mathbb{R}^M$, and $\mathbf{v} \in \mathbb{R}^N$.
- At an optimum, $u_i = \max\{0, 1 - \mathbf{a}^T\mathbf{x}_i + b\}$ and $v_i = \max\{0, 1 + \mathbf{a}^T\mathbf{y}_i - b\}$.
- It can be interpreted as a heuristic for minimizing the number of misclassified points.



Support Vector Machine

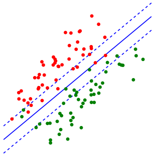
A *support vector machine* (SVM) (sometimes called a *support vector classifier*) represents a trade-off between the width of the slab $\left\{\mathbf{z} : -1 \leq \mathbf{a}^T\mathbf{z} - b \leq 1\right\}$ (given by $2/\left\|\mathbf{a}\right\|_2$) and the number of misclassified points. Specifically, an SVM combines robust linear discrimination with the approximate linear separation heuristic described above.

Support Vector Machine (SVM) Problem:

$$\begin{array}{ll} \text{minimize} & \left\|\mathbf{a}\right\|_2 + \gamma\left(\mathbf{1}^T\mathbf{u} + \mathbf{1}^T\mathbf{v}\right) \\ \text{subject to} & \mathbf{a}^T\mathbf{x}_i - b \geq 1 - u_i \, , \; i = 1, \ldots, M \\ & \mathbf{a}^T\mathbf{y}_i - b \leq -\left(1 - v_i\right) \, , \; i = 1, \ldots, N \\ & \mathbf{u} \succeq \mathbf{0} \, , \; \mathbf{v} \succeq \mathbf{0} \end{array} .$$

- Here, $\gamma > 0$ is the trade-off parameter between slab width and # of misclassified points.
- The solution produces a point on the optimal trade-off curve between inverse of margin $2/\left\|\mathbf{a}\right\|_2$ and classification error, measured by the total slack $\mathbf{1}^T\mathbf{u} + \mathbf{1}^T\mathbf{v}$.

Same example data as in previous slide, with $\gamma = 0.1$:



Linear Discrimination and Logistic Modeling

Another approach to approximately classify the two sets of points when they cannot be linearly separated is based on the *logistic model*. Specifically, we fit the sets of points to a logistic model. Let $\mathrm{z} \in \{0, 1\}$ be a random variable with

$$\begin{aligned}\Pr\{\mathrm{z} = 0\} &= 1/\left(1 + \exp\left(\mathbf{a}^T\mathbf{u} - b\right)\right) , \\ \Pr\{\mathrm{z} = 1\} &= \left(\exp\left(\mathbf{a}^T\mathbf{u} - b\right)\right) / \left(1 + \exp\left(\mathbf{a}^T\mathbf{u} - b\right)\right) ,\end{aligned}$$

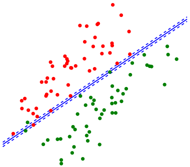
where $\mathbf{u} \in \mathbb{R}^n$ is the explanatory variable of the logistic model. Then, $\{\mathbf{x}_1, \ldots, \mathbf{x}_M\}$ and $\{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$ are the respective values of $\mathbf{u}$ for the M and N samples for which $\mathrm{z} = 0$ and $\mathrm{z} = 1$. We can then determine $\mathbf{a}$ and b using ML estimation, by solving the convex optimization problem

$$\text{minimize} \quad -\ell(\mathbf{a}, b) \quad ,$$

with variables $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, where ℓ is the log-likelihood function

$$\ell(\mathbf{a}, b) = -\sum_{i=1}^{M} \log\left(1 + \exp\left(\mathbf{a}^T\mathbf{x}_i - b\right)\right) + \sum_{i=1}^{N}\left(\mathbf{a}^T\mathbf{y}_i - b\right) - \sum_{i=1}^{N} \log\left(1 + \exp\left(\mathbf{a}^T\mathbf{y}_i - b\right)\right) .$$

- If the data points truly come from a logistic model, then the affine classifier $f(\mathbf{z}) = (\mathbf{a}^\star)^T\mathbf{z} - b^\star$ has the smallest probability of misclassification among all affine classifiers.
- The hyperplane $\mathbf{a}^T\mathbf{u} = b$ corresponds to the points where $\Pr\{\mathrm{z} = 0\} = \Pr\{\mathrm{z} = 1\} = 1/2$, i.e, the two outcomes are equally likely.



Introduction to Nonlinear Discrimination

In many cases, we can separate two sets of points $\{\mathbf{x}_1, \ldots, \mathbf{x}_M\}$ and $\{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$ by a *nonlinear* function $f : \mathbb{R}^n \to \mathbb{R}$ as follows.

$$f(\mathbf{x}_i) > 0\,,\ i = 1, \ldots, M\,,\ f(\mathbf{y}_i) < 0\,,\ i = 1, \ldots, N\,.$$

Linearly parameterized family of functions:

- One way to do this is to choose a linearly parametrized family of functions

  $$f(\mathbf{z}) = \boldsymbol{\theta}^T \mathbf{f}(\mathbf{z})\ ,$$

  where $\boldsymbol{\theta} \in \mathbb{R}^k$ is a parameter vector and $\mathbf{f}(\mathbf{z}) \triangleq (f_1(\mathbf{z})\,, \ldots, f_k(\mathbf{z})) : \mathbb{R}^n \to \mathbb{R}^k$ is a vector of *basis functions*.
- With this linearly parametrized construction, the above inequalities can be solved in the exactly the same way as in linear discrimination.

  Specifically, as f is homogeneous in $\boldsymbol{\theta}$, we solve a set of affine inequalities in $\boldsymbol{\theta}$:

  $$\boldsymbol{\theta}^T \mathbf{f}(\mathbf{x}_i) \geq 1\,,\ i = 1, \ldots, M\,,\ \boldsymbol{\theta}^T \mathbf{f}(\mathbf{y}_i) \leq -1\,,\ i = 1, \ldots, N\,.$$

Quadratic Discrimination

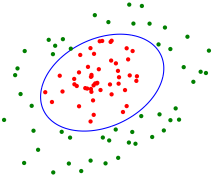
Suppose that $f(\mathbf{z}) = \mathbf{z}^T \mathbf{P} \mathbf{z} + \mathbf{q}^T \mathbf{z} + r$, where the parameters to be chosen are $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{q} \in \mathbb{R}^n$, and $r \in \mathbb{R}$. Here, we must satisfy the affine inequalities

$$\mathbf{x}_i^T \mathbf{P} \mathbf{x}_i + \mathbf{q}^T \mathbf{x}_i + r \geq 1 \,,\ i = 1, \ldots, M \,,\ \mathbf{y}_i^T \mathbf{P} \mathbf{y}_i + \mathbf{q}^T \mathbf{y}_i + r \leq -1 \,,\ i = 1, \ldots, N \,.$$

We can also pose additional constraints. For example, we can require $\mathbf{P} \prec \mathbf{0}$, which means that the separating surface is ellipsoidal. In other words, we seek an ellipsoid containing all the points $\{\mathbf{x}_1, \ldots, \mathbf{x}_M\}$, but none of the points $\{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$. This quadratic discrimination problem can then be solved as an SDP feasibility problem:

$$\begin{array}{ll} \text{find} & \mathbf{P}, \mathbf{q}, r \\ \text{subject to} & \mathbf{x}_i^T \mathbf{P} \mathbf{x}_i + \mathbf{q}^T \mathbf{x}_i + r \geq 1 \,,\ i = 1, \ldots, M \\ & \mathbf{y}_i^T \mathbf{P} \mathbf{y}_i + \mathbf{q}^T \mathbf{y}_i + r \leq -1 \,,\ i = 1, \ldots, N \\ & \mathbf{P} \preceq -\mathbf{I} \end{array} ,$$

with variables $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{q} \in \mathbb{R}^n$, and $r \in \mathbb{R}$. (Here, we use homogeneity in $\mathbf{P}$, $\mathbf{q}$, and r to express the constraint $\mathbf{P} \prec \mathbf{0}$ as $\mathbf{P} \preceq -\mathbf{I}$.)



Polynomial Discrimination

Consider the set of polynomials on $\mathbb{R}^n$ with degree less than or equal to d:

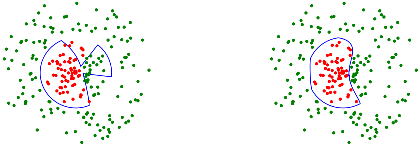
$$f(\mathbf{z}) = \sum_{i_1+\cdots+i_n \le d} a_{i_1,\ldots,i_d} z_1^{i_1} \cdots z_n^{i_n} \,.$$

Note that this can be expressed as $f(\mathbf{z}) = \boldsymbol{\theta}^T \mathbf{f}(\mathbf{z})$, where $\boldsymbol{\theta}$ is a vector of all the coefficients $a_{i_1,\ldots,i_d}$ and $\mathbf{f}(\mathbf{z})$ consists of all monomials up to the given degree of d.

- By solving the discrimination feasibility problem

$$\begin{array}{ll} \text{find} & \boldsymbol{\theta} \\ \text{subject to} & \boldsymbol{\theta}^T \mathbf{f}(\mathbf{x}_i) \ge 1 \,,\ i = 1, \ldots, M \\ & \boldsymbol{\theta}^T \mathbf{f}(\mathbf{y}_i) \le -1 \,,\ i = 1, \ldots, N \end{array} \,,$$

  we check, geometrically, whether the two sets can be separated by an algebraic surface.
- As the degree is a quasiconvex function of the coefficients, we can find the minimum degree polynomial on $\mathbb{R}^n$ that separates two sets of points via quasiconvex programming.



No cubic polynomial can separate the points (left), whereas a quartic polynomial can do so (right).

Introduction to Placement and Location Problems

- We are given N points with coordinates $\mathbf{x}_i \in \mathbb{R}^k$, where $k = 2$ or $k = 3$.
- Some of the locations $\mathbf{x}_i$ are given, while the other positions $\mathbf{x}_i$ are variables.
- For each pair of points, we associate a cost function $f_{i,j}(\mathbf{x}_i, \mathbf{x}_j)$.

Placement Problem:

$$\text{minimize} \quad \sum_{(i,j)\in\mathcal{A}} f_{i,j}(\mathbf{x}_i, \mathbf{x}_j) \quad .$$

- Here, $\mathcal{A}$ is the set of all links in the graph described by the N points or nodes.
- Alternatively, the objective can be expressed as a sum over all arcs (i, j) if we set $f_{i,j} = 0$ when links i and j are not connected.
- The variables are the positions of the free points.

Interpretations:

- The points may represent plants or warehouses; $f_{i,j}$ could be the transportation cost between facilities i and j.
- The points may represent cells on an integrated circuit (IC); $f_{i,j}$ could then represent the wirelength.

Linear Facility Location Problems

The simplest version of the placement problem occurs when the cost function $f_{i,j}$ is a linear function of the distance between nodes i and j. In other words, we have

$$f_{i,j}(\mathbf{x}_i, \mathbf{x}_j) = w_{i,j} \left|\left|\mathbf{x}_i - \mathbf{x}_j\right|\right| \ ,$$

where $w_{i,j} \geq 0$ is a weight parameter. By proper choice of the weights $w_{i,j}$, the placement problem objective can be expressed in the following two ways.

$$\sum_{(i,j)\in\mathcal{A}} w_{i,j} \left|\left|\mathbf{x}_i - \mathbf{x}_j\right|\right| = \sum_{i<j} w_{i,j} \left|\left|\mathbf{x}_i - \mathbf{x}_j\right|\right| \ .$$

For this choice of cost function, the placement problem is always convex.

Examples: *One free point* $(u, v) \in \mathbb{R}^2$ *with fixed points* $(u_1, v_1), \ldots, (u_K, v_K)$*.*

- ℓ_1*-norm:* When $w_{i,j} = 1$ for all i, j, the objective is

  $$\sum_{i=1}^{K} \left(|u - u_i| + |v - v_i|\right) \ .$$

  An optimal point is any *median* of the fixed points. That is, u and v can be taken to be any median of the points $\{u_1, \ldots, u_K\}$ and $\{v_1, \ldots, v_K\}$, respectively.
- *Euclidean norm:* When $w_{i,j} = 1$ for all i, j, the objective is

  $$\sum_{i=1}^{K} \left((u - u_i)^2 + (v - v_i)^2\right)^{1/2} \ .$$

  The point (u, v) that minimizes this is called the *geometric median* or the *Fermat-Weber point* of the given fixed points.

Types of Placement Constraints

There are several interesting constraints that can be included in the placement problem which preserve convexity.

- We can require some positions $\mathbf{x}_i$ to lie in a specified convex set, e.g., a particular line, interval, square, or ellipsoid.
- The relative position of one point with respect to one or more other points can be constrained. For example, we can limit the distance between a pair of points.
- We can impose relative position constraints, e.g., that one point must lie to the left of another point.
- Several *bounding box* constraints can be included. The bounding box of a group of points is the smallest rectangle that contains the points. For example, we can constrain the points $\mathbf{x}_1, \ldots, \mathbf{x}_p$ to lie in a bounding box with perimeter not exceeding $P_{\max}$ by adding the constraints

$$\mathbf{u} \preceq \mathbf{x}_i \preceq \mathbf{v}\,,\ i = 1, \ldots, p\,,\ 2\mathbf{1}^T(\mathbf{v} - \mathbf{u}) \leq P_{\max}\,,$$

where $\mathbf{u} \in \mathbb{R}^k$ and $\mathbf{v} \in \mathbb{R}^k$ are additional variables.

Nonlinear Facility Location Problems

A generalization of the linear facility location problem occurs when we take the cost function to be

$$f_{i,j}(\mathbf{x}_i, \mathbf{x}_j) = w_{i,j} h(||\mathbf{x}_i - \mathbf{x}_j||) \ ,$$

where $w_{i,j} \geq 0$ is a weight parameter and h is a nondecreasing (on $\mathbb{R}_+$) and convex function. In this case, we call the associated problem a *nonlinear placement* or *facility location problem*.

Examples:

- *Quadratic placement problem:* When the Euclidean norm is used with $h(z) = z^2$, we obtain the *quadratic placement problem*. As an example, when there is one free point $\mathbf{x}$ and fixed points $\mathbf{x}_1, \ldots, \mathbf{x}_K$, and $w_{i,j} = 1$ for all i, j, the placement problem becomes

  $$\text{minimize} \quad ||\mathbf{x} - \mathbf{x}_1||_2^2 + \cdots + ||\mathbf{x} - \mathbf{x}_K||_2^2 \quad .$$

  This has the optimal solution $\mathbf{x}^\star = (\mathbf{x}_1 + \cdots + \mathbf{x}_K)/K$, which is the average or centroid of the fixed points.
- *Power function:* (with power $p \geq 1$)

  $$h(z) = z^p \, .$$

- *Deadzone-linear function:* (with deadzone width 2γ)

  $$h(z) = \begin{cases} 0 \, , & |z| \leq \gamma \\ |z - \gamma| \ , & |z| > \gamma \end{cases} \quad .$$

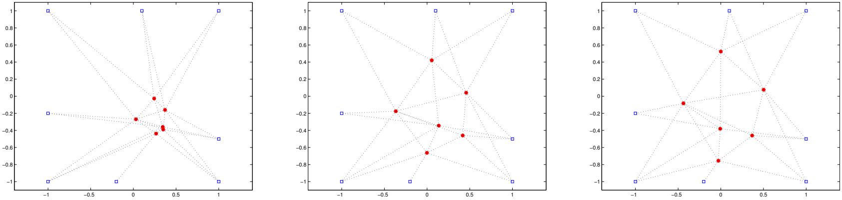
- *Quadratic-linear function:* (with quadratic width 2γ)

  $$h(z) = \begin{cases} z^2 \, , & |z| \leq \gamma \\ 2\gamma |z| - \gamma^2 \, , & |z| > \gamma \end{cases} \quad .$$

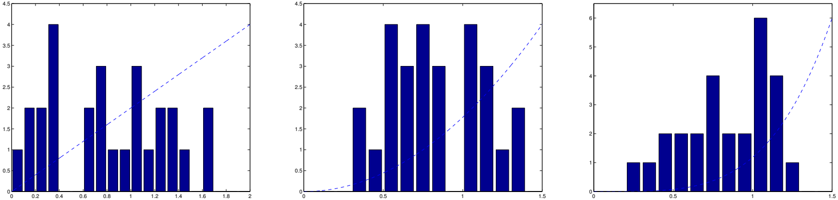
Comparison Between Linear and Nonlinear Placement

Example: Minimize $\sum_{(i,j)\in\mathcal{A}} h\big(||\mathbf{x}_i - \mathbf{x}_j||_2\big)$, with 6 free points (red dots), 8 fixed points (blue squares), and 27 links (black dotted lines).

Optimal placement for $h(z) = z$, $h(z) = z^2$, and $h(z) = z^4$ (from left to right, respectively).



Histograms of respective connection lengths $||\mathbf{x}_i - \mathbf{x}_j||_2$ for $h(z) = z$, $h(z) = z^2$, and $h(z) = z^4$.



Path Constraints

A p-link *path* along the points $\mathbf{x}_1, \ldots, \mathbf{x}_N$ is described by a sequence of nodes, $i_0, \ldots, i_p \in \{1, \ldots, N\}$. The length of the path is given by

$$\left|\left|\mathbf{x}_{i_1} - \mathbf{x}_{i_0}\right|\right| + \left|\left|\mathbf{x}_{i_2} - \mathbf{x}_{i_1}\right|\right| + \cdots + \left|\left|\mathbf{x}_{i_p} - \mathbf{x}_{i_{p-1}}\right|\right| .$$

- The path length is a convex function of $\mathbf{x}_1, \ldots, \mathbf{x}_N$.
- As a result of the convexity of the path length, imposing an upper bound on the length of a path is a convex constraint.
- Several interesting placement problems involve path constraints, or have an objective based on path lengths.
- One typical example is the *minimax delay placement problem*, in which the objective is based on a maximum path length over a set of paths.

Minimax Delay Placement Problem

To introduce the minimax delay placement problem, we first describe the topology characterizing the problem.

- We consider a directed, acyclic graph with nodes $1, \ldots, N$, and arcs or links represented by a set $\mathcal{A}$ of ordered pairs: $(i, j) \in \mathcal{A}$ if and only if an arc points from i to j.
- Node i is a *source node* if no arc $\mathcal{A}$ points to it; it is a *sink node* or *destination node* if no arc in $\mathcal{A}$ leaves from it.
- The arcs model a kind of flow, say of goods or information, in a network with nodes at positions $\mathbf{x}_1, \ldots, \mathbf{x}_N$. The flow starts at a source node, moves along a path from node to node, and ends up at a sink or destination node.
- We use the distance between successive nodes to model propagation time, or shipment time, of goods between nodes; the total delay or propagation time of a path is then proportional to the sum of the distances between successive nodes.
- Some node locations are fixed, while others are free. The goal is to choose the free node locations to minimize the maximum total delay, for any path from a source node to a sink node. This leads to a convex problem, since the objective

$$T_{\max} \triangleq \max\left\{ ||\mathbf{x}_{i_1} - \mathbf{x}_{i_0}|| + \cdots + \left|\left|\mathbf{x}_{i_p} - \mathbf{x}_{i_{p-1}}\right|\right| : i_0, \ldots, i_p \text{ is a source-sink path} \right\} ,$$

  is a convex function of the locations $\mathbf{x}_1, \ldots, \mathbf{x}_N$.

Minimax Delay Placement Problem (Continued)

- While the minimax delay placement problem is convex, the number of source-sink paths is exponential in the number of nodes or arcs, which can become quite large.
- However, the problem can be reformulated to avoid enumerating all sink-source paths.
- To show this, let τ_k denote the maximum total delay of any path from node k to a sink node. Suppose node k has outgoing arcs to nodes $j_1, \ldots, j_p$. Then, based on a simple *dynamic programming* argument, we can show that

$$\tau_k = \max\left\{||\mathbf{x}_{j_1} - \mathbf{x}_k|| + \tau_{j_1}, \ldots, \left|\left|\mathbf{x}_{j_p} - \mathbf{x}_k\right|\right| + \tau_{j_p}\right\} .$$

- The above equation gives a recursion for finding the maximum delay from any node. Specifically, we start at the sink nodes, and work backward until we reach all source nodes.
- The number of arithmetic operations required for this is approximately the number of links.
- With this recursion, the minimax delay placement problem can be expressed as

$$\begin{array}{ll} \text{minimize} & \max\{\tau_k : k \text{ is a source node}\} \\ \text{subject to} & \tau_k = 0\,,\ k \text{ is a sink node} \\ & \tau_k = \max\{||\mathbf{x}_j - \mathbf{x}_k|| + \tau_j : \text{there is an arc from } k \text{ to } j\} \end{array} .$$

  This form of the problem, however, is not convex.
- By replacing the equality constraints with inequalities, we obtain an equivalent form of the problem that is convex. Introducing new variables $T_1, \ldots, T_N$ as upper bounds on $\tau_1, \ldots, \tau_N$, the minimax delay placement problem can be expressed as the convex problem,

$$\begin{array}{ll} \text{minimize} & \max\{T_k : k \text{ is a source node}\} \\ \text{subject to} & T_k = 0\,,\ k \text{ is a sink node} \\ & T_k \geq \max\{||\mathbf{x}_j - \mathbf{x}_k|| + T_j : \text{there is an arc from } k \text{ to } j\} \end{array} .$$

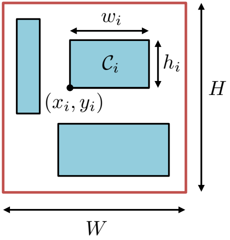
Introduction to Floor Planning

A *floor planning problem* is an extension of a placement problem in the following ways:

- The objects to be placed are rectangles or boxes aligned with the axes (as opposed to points), and must not overlap.
- Each rectangle or box to be placed can be reconfigured, within some limits.

Typically, the objective is to minimize the size (e.g., area, volume, or perimeter) of the *bounding box*, i.e., the smallest box containing the boxes to be configured and placed.

- The non-overlap constraints make the general floor planning problem a complicated combinatorial optimization problem or rectangular packing problem.
- However, if the *relative positioning* of the boxes is specified, then several types of floor planning problems can be formulated as convex optimization problems.



Introduction to Floor Planning (Continued)

We will focus on the two-dimensional case of floor planning here.

Assumptions and required constraints:

- We have N cells or modules $\mathcal{C}_1, \ldots, \mathcal{C}_N$ that are to be configured and placed in a rectangle with width W and height H, and lower left corner at the origin $(0, 0)$.
- The geometry and position of the i-th cell is specified by its width w_i and height h_i, and the coordinates (x_i, y_i) of its lower left corner.
- The variables of the problem are x_i, y_i, h_i, w_i for $i = 1, \ldots, N$, and the width W and height H of the bounding rectangle.
- We require the cells to lie inside the bounding rectangle, i.e.,

$$x_i \geq 0\,,\ y_i \geq 0\,,\ x_i + w_i \leq W\,,\ y_i + h_i \leq H\,,\ i = 1, \ldots, N\,.$$

- We also require that the cells do not overlap, expect possibly on their boundaries:

$$\mathrm{int}(\mathcal{C}_i \cap \mathcal{C}_j) = \varnothing \text{ for } i \neq j\,.$$

(A minimum clearance can be included as well.) This holds if and only if, for $i \neq j$,

$\mathcal{C}_i$ is left of $\mathcal{C}_j$, or $\mathcal{C}_i$ is right of $\mathcal{C}_j$, or $\mathcal{C}_i$ is below $\mathcal{C}_j$, or $\mathcal{C}_i$ is above $\mathcal{C}_j$.

These geometric conditions correspond to the following inequalities, for $i \neq j$.

$$x_i + w_i \leq x_j\,, \text{ or } x_j + w_j \leq x_i\,, \text{ or } y_i + h_i \leq y_j\,, \text{ or } y_j + h_j \leq y_i\,.$$

Relative Positioning Constraints

To specify relative positioning constraints, we define relations $\mathcal{L}$ and $\mathcal{B}$ on pairs of nodes which specify whether cells are to the left of or below each other, respectively.

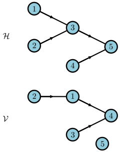
- We have that $\mathcal{C}_i$ is to the left of $\mathcal{C}_j$ if $(i, j) \in \mathcal{L}$, and similarly $\mathcal{C}_i$ is below $\mathcal{C}_j$ if $(i, j) \in \mathcal{B}$.
- This yields the following constraints for $i, j = 1, \ldots, N$.

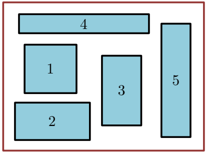
  $$x_i + w_i \leq x_j \text{ for } (i, j) \in \mathcal{L}\,,\ y_i + h_i \leq y_j \text{ for } (i, j) \in \mathcal{B}\,.$$

  We require that for each (i, j) with $i \neq j$, one of the following must hold:

  $$(i, j) \in \mathcal{L}\,,\ (j, i) \in \mathcal{L}\,,\ (i, j) \in \mathcal{B}\,,\ (j, i) \in \mathcal{B}\,.$$

  In addition, $(i, i) \notin \mathcal{L}$ and $(i, i) \notin \mathcal{B}$. This leads to a set of $N(N-1)/2$ affine inequalities.
- The relations $\mathcal{L}$ and $\mathcal{B}$ are anti-symmetric (i.e., $(i, j) \in \mathcal{L} \Rightarrow (j, i) \notin \mathcal{L}$) and transitive (i.e., $(i, j) \in \mathcal{L}\,,\ (j, k) \in \mathcal{L} \Rightarrow (i, k) \in \mathcal{L}$). Transitivity leads to removing redundant constraints.
- A minimal set of relative positioning constraints can be described using two directed acyclic graphs $\mathcal{H}$ (for horizontal) and $\mathcal{V}$ (for vertical). We have $(i, j) \in \mathcal{L}$ (similarly $(i, j) \in \mathcal{B}$) if and only if there is a directed path in $\mathcal{H}$ (similarly $\mathcal{V}$) from i to j.
- To ensure a relative positioning constraint for every pair of cells, we require that for every pair, there must be a directed path from one cell to the other in one of the graphs.





Relative Positioning Constraints (Continued)

- We only need to impose the inequalities corresponding to the edges of the graphs $\mathcal{H}$ and $\mathcal{V}$; the others follow from transitivity.
- So, for relative positioning, we need only enforce the following set of inequalities.

$$x_i + w_i \le x_j \text{ for } (i,j) \in \mathcal{H}\,,\ y_i + h_i \le y_j \text{ for } (i,j) \in \mathcal{V}\,.$$

- In a similar way, the $4N$ bounding box inequalities,

$$x_i \ge 0\,,\ y_i \ge 0\,,\ x_i + w_i \le W\,,\ y_i + h_i \le H\,,\ i = 1, \ldots, N\,,$$

  can be reduced to a minimal equivalent set.
- For example, the constraint $x_i \ge 0$ need only be imposed on the left-most cells. Similarly, the inequalities $x_i + w_i \ge W$ need only be enforced for the right-most cells. Analogous results hold for the vertical bounding box inequalities.
- This leads to the following minimal equivalent set of bounding box inequalities.

$$\begin{aligned} x_i \ge 0 \text{ for } i\ \mathcal{L} \text{ minimal}, &\quad x_i + w_i \le W \text{ for } i\ \mathcal{L} \text{ maximal}, \\ y_i \ge 0 \text{ for } i\ \mathcal{B} \text{ minimal}, &\quad y_i + h_i \le H \text{ for } i\ \mathcal{B} \text{ maximal}. \end{aligned}$$

For the example above, a minimal set of horizontal relative positioning inequalities is given by

$$\begin{gathered} x_1 \ge 0\,,\ x_2 \ge 0\,,\ x_4 \ge 0\,,\ x_5 + w_5 \le W\,,\ x_1 + w_1 \le x_3\,, \\ x_2 + w_2 \le x_3\,,\ x_3 + w_3 \le x_5\,,\ x_4 + w_4 \le x_5\,. \end{gathered}$$

Similarly, a minimal set of vertical relative positioning inequalities is given by

$$\begin{gathered} y_2 \ge 0\,,\ y_3 \ge 0\,,\ y_5 \ge 0\,,\ y_4 + h_4 \le H\,,\ y_5 + h_5 \le H\,, \\ y_2 + h_2 \le y_1\,,\ y_1 + h_1 \le y_4\,,\ y_3 + h_3 \le y_4\,. \end{gathered}$$

Floor Planning Convex Constraints

For all floor planning problems considered here, we will take the objective to be the perimeter of the bounding box, i.e., $2(W+H)$, which is a linear function of the variables. A list of several convex constraints that can be added to the floor planning problem in addition to the convex relative positioning and bounding box constraints is as follows.

- *Minimum spacing:* ($\rho^{\mathrm{h}}_{i,j} \geq 0, \rho^{\mathrm{v}}_{i,j} \geq 0$)

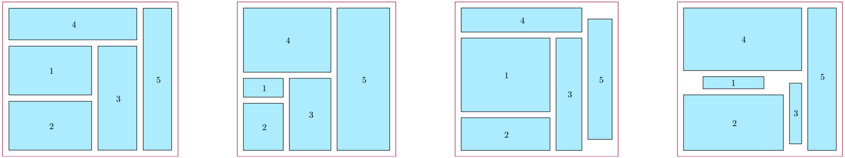
$$x_i + w_i + \rho^{\mathrm{h}}_{i,j} \leq x_j \text{ for } (i,j) \in \mathcal{H}\ , \text{ and/or } y_i + h_i + \rho^{\mathrm{v}}_{i,j} \leq y_j \text{ for } (i,j) \in \mathcal{V}\,.$$

- *Minimum cell area:* ($A_i \geq 0$)

$$w_i h_i \geq A_i \iff w_i \geq A_i/h_i \iff (w_i h_i)^{1/2} \geq A_i^{1/2} \iff \log w_i + \log h_i \geq \log A_i\,.$$

- *Aspect ratio constraints:* ($u_i \geq l_i \geq 0$)

$$l_i \leq w_i/h_i \leq u_i\,.$$



Examples using above ordering constraints, with the same minimum required spacing of $\rho = 1$, and the same aspect ratio constraint $1/5 \leq w_i/h_i \leq 5$, but different minimum required cell areas A_i. Here, the values of A_i are chosen such that $\sum_{i=1}^{5} A_i$ is the same for each case.

Floor Planning Convex Constraints (Continued)

- *Alignment constraints:* We can impose that a cell is aligned with another cell or the boundary box in a certain way. All such conditions are affine equality constraints.
- *Symmetry constraints:* We can require pairs of cells to be symmetric about a vertical or horizontal axis, that can be fixed or floating. This leads to affine equality constraints.
- *Similarity constraints:* We can impose the width and/or height of cell i to be a scale factor, say a, of cell j, by imposing the linear equality constraints $w_i = aw_j$ and/or $h_i = ah_j$.
- *Containment constraints:* We can require that a particular cell contains a given point or lies inside a given polyhedron, by imposing affine inequality constraints.
- *Distance constraints:* Several convex distance constraints can be imposed. For example, to limit the distance between the centers of cells i and j, we use the convex inequality

$$||(x_i + w_i/2, y_i + h_i/2) - (x_j + w_j/2, y_j + h_j/2)|| \leq D_{i,j} \, .$$

  We can also require $\mathrm{dist}(\mathcal{C}_i, \mathcal{C}_j) \leq D_{i,j}$, by introducing four new variables u_i, v_i, u_j, v_j, and imposing the affine inequalities

$$x_i \leq u_i \leq x_i + w_i \, , \; y_i \leq v_i \leq y_i + h_i \, ,$$

  along with the convex inequality

$$||(u_i, v_i) - (u_j, v_j)|| \leq D_{i,j} \, .$$

  - *ℓ_∞-norm:* We have $\mathrm{dist}(\mathcal{C}_i, \mathcal{C}_j) \leq D_{i,j}$ if and only if $x_j - (x_i + w_i) \leq D_{i,j}$, $y_j - (y_i + h_i) \leq D_{i,j}$, and $y_i - (y_j + h_j) \leq D_{i,j}$.
  - *ℓ_1-norm or ℓ_2-norm:* We have $\mathrm{dist}(\mathcal{C}_i, \mathcal{C}_j) \leq D_{i,j}$ if and only if $y_j - (y_i + h_i) \leq d_v$, $y_i - (y_j + h_j) \leq d_v$, $d_v \geq 0$, and $x_j - (x_i + w_i) + d_v \leq D_{i,j}$ for the ℓ_1-norm or $(x_j - (x_i + w_i))^2 + d_v^2 \leq D_{i,j}^2$ for the ℓ_2-norm.

Floor Planning Via Geometric Programming

The floor planning problem can also be formulated as a GP in the variables x_i, y_i, w_i, h_i, W, H.

Relative positioning and bounding box constraints:

- The relative positioning constraints can be expressed in the posynomial form,
$$x_i x_j^{-1} + w_i x_j^{-1} \leq 1 \text{ for } (i,j) \in \mathcal{H}\,,\ y_i y_j^{-1} + h_i y_j^{-1} \leq 1 \text{ for } (i,j) \in \mathcal{V}\,.$$
- The bounding box constraints have the following posynomial form.
$$x_i W^{-1} + w_i W^{-1} \leq 1 \text{ for } i\ \mathcal{L} \text{ maximal}\,,\ y_i H^{-1} + h_i H^{-1} \leq 1 \text{ for } i\ \mathcal{B} \text{ maximal}\,.$$
Note that here, the constraints that $x_i \geq 0$ for i $\mathcal{L}$ minimal and $y_i \geq 0$ for i $\mathcal{B}$ minimal are implicit as a result of the domain of the GP problem formulation.

Nuances of GP formulation of floor planning problem:

- We can minimize the bounding box area WH, since it is a posynomial.
- We can exactly specify the area of any cell, since $w_i h_i = A_i$ is a monomial equality constraint.
- Alignment, symmetry, and distance constraints cannot be handled with the GP formulation.
- Similarity can be handled with the GP formulation, where the scaling ratio can either be given or unspecified (in which case it can be treated as another variable).