EE/ACM 150 - Applications of Convex Optimization in Signal Processing and Communications
Lecture 1

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Outline

Brief Overview of General Optimization Problems

General Optimization Problem

$$\begin{array}{ll} \text{minimize} & f_0(\mathbf{x}) \\ \text{subject to} & f_k(\mathbf{x}) \leq 0\,,\ k = 1, \ldots, m \\ & h_\ell(\mathbf{x}) = 0\,,\ \ell = 1, \ldots, p \end{array} .$$

$$\begin{array}{rcl} \mathbf{x} \in \mathbb{R}^n & - & \text{optimization variable} \\ f_0(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R} & - & \text{objective (or cost) function} \\ f_k(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R} & - & k\text{-th inequality constraint function} \\ h_\ell(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R} & - & \ell\text{-th equality constraint function} \end{array}$$

An **optimal solution** $\mathbf{x}^\star$ is one for which the value of f_0 is smallest among all vectors that satisfy the constraints.

Brief Overview of Convex Optimization Problems

Convex Optimization Problem

$$\begin{array}{ll} \text{minimize} & f_0(\mathbf{x}) \\ \text{subject to} & f_k(\mathbf{x}) \leq 0\,,\ k = 1, \ldots, m \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{array} .$$

- $f_0(\mathbf{x})\,, \{f_k(\mathbf{x})\}$ are **convex functions.**
- $\mathbf{A} \in \mathbb{R}^{p \times n}, \mathbf{b} \in \mathbb{R}^p$ define the **affine** equality constraint vector function $\mathbf{h}(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b}$.

Convex function: A function $g : \mathbb{R}^n \to \mathbb{R}$ is **convex** if $\mathrm{dom}\,\{g\}$ is a **convex set** and if, for all $\mathbf{x}, \mathbf{y} \in \mathrm{dom}\,\{g\}$ and θ with $0 \leq \theta \leq 1$, we have

$$g(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta g(\mathbf{x}) + (1-\theta)g(\mathbf{y}) \ .$$

Why Study Convex Optimization?

- Many interesting and important problems in signal processing, communications, statistics, machine learning, medical imaging, and finance can be posed as such.
- It represents a generalization of several least-squares results you may already be aware of.
- Polynomial-time (P) algorithms exist to *globally* solve such problems numerically.
  - Computation time of interior-point methods roughly proportional to $\max\left\{n^3, n^2 m, F\right\}$, where F denotes the cost of evaluating $\{f_k(\mathbf{x})\}$ as well as their first and second derivatives.
  - We are *guaranteed* to have a *certificate* of optimality or infeasibility.
- Convex optimization techniques can be used to provide good heuristic suboptimal solutions and/or bounds to nonconvex problems believed to be non-deterministic polynomial-time (NP) complete or hard.

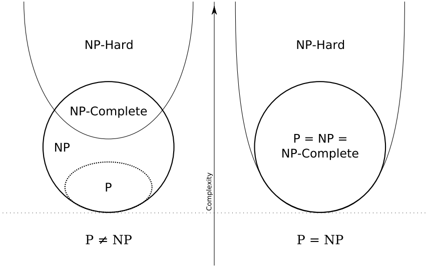
Commentary on Computational Complexity

Pertaining to the unsolved problem in computer science of whether P = NP or P $\neq$ NP

A decision problem $\mathcal{C}$ is NP-complete if:

1. $\mathcal{C} \in$ NP,
2. Every problem in NP is reducible to $\mathcal{C}$ in polynomial time.

Any problem which satisfies condition 2 is called NP-hard.



- All convex optimization problems we consider have P complexity.
- Convex optimization techniques can be used to provide heuristic bounds (and often suboptimal solutions) to NP-hard problems.

Brief History of Contributions to the Field

Contribution	Date
Dantzig introduces simplex algorithm for solving linear programming (LP) problems	1947
Wiener develops Wiener filter which is statistically optimal in a minimum mean square error (MMSE) sense	1949
Early work on interior-point methods documented in landmark book by Fiacco and McCormick	1968
Shor develops the ellipsoid method and Khachiyan uses it to show polynomial-time solvability of LPs	1972, 1979
Karmarkar develops polynomial-time interior-point method for solving LPs	1984
Mehrotra develops predictor-corrector algorithm for solving LPs, forming the basis of primal-dual interior-point methods	1989, 1992
Nesterov and Nemirovski develop polynomial-time interior-point methods for solving nonlinear convex optimization problems	1994
Grant, Boyd, and Ye introduce `cvx`, a freely available MATLAB-based software modeling system for disciplined convex programming (DCP)	2006

What This Course Is...

- One of the goals is to expose the applicability of convex optimization techniques to a wide variety of different fields.
- Another objective is to introduce the "*tricks of the trade*" to apply to several design problems in order to:
  - express them in standard convex forms (when possible),
  - use convex optimization techniques to provide suitable heuristics (when impossible).
- In addition, one key element is to familiarize the students with readily available software packages for convex optimization (most notably `cvx` for MATLAB).
- Finally, yet another aim is to dispel the fear of non-least-squares optimization (i.e., to possibly make you ℓ_2-converts).

What This Course Isn't...

- Unlike a traditional course solely on convex optimization, we will try not to dwell too much on theory.
- In addition, we will cover neither algorithmic nor implementation aspects of interior-point methods used to solve convex optimization problems.
- Finally, we will not cover some of the more modern subjects such as:
  - subgradient methods used to solve nondifferentiable convex problems as well as large scale problems,
  - decentralized and distributed methods for solving convex optimization problems.

Course Syllabus

General overview of lecture topics

Lecture #	Topic	Date	Notes
1	Introduction	4/3	
2	Linear algebra review	4/5	HW #1 out
3	Singular value decomposition (SVD), Moore-Penrose pseudoinverse	4/10	
4	Special vector/matrix operators, the Schur complement	4/12	HW #1 due, HW #2 out
5	Matrix calculus concepts (real/complex matrix differentiation)	4/17	
6	Convex sets, generalized inequalities	4/19	HW #2 due, HW #3 out
7	Convex functions, quasiconvex functions, log-concave/convex functions	4/24	
8	Introduction to convex optimization problems: LPs, QPs, QCQPs, & SOCPs	4/26	HW #3 due, HW #4 out
9	Introduction to convex optimization problems: GPs & SDPs	5/1	
10	Vector optimization problems, Pareto optimal points, scalarization	5/3	HW #4 due, HW #5 out
11	Duality: Lagrange dual function/problem, weak & strong duality, geometric interpretations	5/8	
12	Duality: KKT conditions, perturbation and sensitivity analysis, problems with generalized inequalities	5/10	HW #5 due, HW #6 out
13	Approximation and fitting problems with regularization	5/15	
14	Robust approximation, function fitting and interpolation problems	5/17	HW #6 due, HW #7 out
15	Statistical estimation: parametric/nonparametric estimation, optimal detector design	5/22	
16	Statistical estimation: probability bounds, experiment design	5/24	HW #7 due, HW #8 out
17	Geometric problems: Euclidean distance problems, extremal volume ellipsoids, centering problems	5/29	
18	Geometric problems: classification, placement and location, floor planning	5/31	HW #8 due
19	Conclusion (bonus lecture, focus to be determined (TBD))	6/5	optional lecture

Administrative Details

Lecture Time & Location: Tue. & Thu. 1-2:30 PM, 080 Moore

Course Website URL:
http://www.systems.caltech.edu/dsp/ee150_acospc/

Course E-Mail: ee150.acospc@gmail.com

Teaching Assistants:

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Andre Tkacenko	Chih-Hao (John) Liu	Piya Pal
Mon. & Wed. 3-4 PM	Mon. & Wed. 2-3 PM	Mon. & Wed. 6-7 PM
110 Moore	110 Moore	110 Moore

Course Texts & Supplemental Material

All resources for the class are available online and completely *free*!

Texts:

- *Convex Optimization* by Stephen Boyd and Lieven Vandenberghe
- *The Matrix Cookbook* by Kaare Brandt Petersen and Michael Syskind Pedersen

Supplemental Material:

- **MATLAB Software Package:**
  `cvx` (download link and users' guide) by Michael Grant, Stephen Boyd, and Yinyu Ye
- **Exercises:**
  Additional Exercises for *Convex Optimization* by Stephen Boyd and Lieven Vandenberghe
- **Extra Resources:**
  Complete problems and solutions, as well as other material, for *Convex Optimization Theory* by Dimitri P. Bertsekas

Grading Policies

Grading for the class will be based *entirely* on weekly homeworks. There will be no exams. Barring extenuating circumstances, there will be 8 homework sets. Nominally, each homework set will have one *no collaboration type* (NCT) problem, which will carry more weight than other *limited collaboration type* (LCT) problems. Notionally, each homework set will be 70 points, comprised as follows:

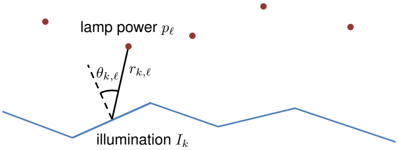
Homework Set Problem	Problem Type	Points
1	LCT	10
2	LCT	10
3	LCT	10
4	LCT	10
*5	NCT	30

Grades will be assigned according to the following (rounded) percentage ranges.

Grade	Percentage
A^-, A, A^+	90-92, 93-96, 97-100
B^-, B, B^+	80-82, 83-86, 87-89
C^-, C, C^+	70-72, 73-76, 77-79
D^-, D, D^+	60-62, 63-66, 67-69
F	0-59

Patch Illumination Example

Suppose we have m lamps illuminating n small, flat patches as shown.



The intensity of the illumination at the k-th patch I_k depends linearly on the lamp powers $\{p_\ell\}$.

$$I_k = \sum_{\ell=1}^{m} a_{k,\ell} p_\ell \, , \; a_{k,\ell} = r_{k,\ell}^{-2} \max \{\cos \theta_{k,\ell}, 0\} \; .$$

The goal is to make each intensity as close to a desired value I_{des} as possible (in some sense), subject to a maximum power constraint $p_{\max}$ for the lamps.

Problem:

$$\begin{array}{ll} \text{minimize} & \max\limits_{k=1,\ldots,n} |\log I_k - \log I_{\text{des}}| \\ \text{subject to} & 0 \le p_\ell \le p_{\max} \, , \; \ell = 1, \ldots, m \end{array} \; .$$

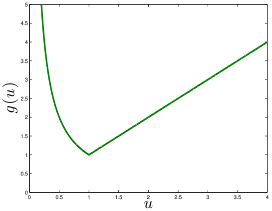
Patch Illumination Example (Continued)

The above problem can be shown to be equivalent to the following one.

Equivalent Problem:

$$\begin{array}{ll} \text{minimize} & f_0(\mathbf{p}) \triangleq \max\limits_{k=1,\ldots,n} g(I_k/I_{\text{des}}) \\ \text{subject to} & 0 \leq p_\ell \leq p_{\max}\,,\ \ell = 1,\ldots,m \end{array} .$$

Here, $g(u) \triangleq e^{|\log u|} = \max\{1/u, u\}$.



It can be shown that $f_0(\mathbf{p})$ is *convex* since the *maximum* of *convex functions* is *convex*. Hence, the equivalent problem is a convex optimization type which can be numerically solved exactly using an interior-point method (with complexity similar to least-squares).

Additional Constraints

Convex or nonconvex

Suppose we add either of the following constraints to the patch illumination problem:

1. no more than half of the total power is in any m_0 lamps,
2. no more than half of the lamps are on (i.e., $p_\ell > 0$).

Questions: How does each complicate the problem? Can each constraint be expressed in convex form or not?

Answers: With (1), the constraint can be expressed in convex form and so the problem is still easy to solve. However with (2), this is not possible and the resulting problem turns out to be very hard to solve.

Moral: To the untrained eye, very simple problems can appear quite similar to very difficult ones.

General Aims of the Course

Overall, the general goals of this class are the following:

- to introduce you to the mathematical tools (linear algebra, matrix calculus, etc.) used to analyze and characterize convex optimization problems,
- to recognize/formulate problems (such as the one considered in the illumination example) as convex optimization problems (whenever possible),
- to advocate and justify the use of convex optimization techniques as heuristics for nonconvex problems,
- to acquaint you with powerful software used to numerically solve convex optimization problems,
- to cover a variety of applications in signal processing and communications in which convex optimization arises.