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Eigenfunctions of the Fourier Transform[1]

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An eigenfunction of the Fourier transform operator is a function whose shape is identical to that of its Fourier transform. The Gaussian curve, appropriately scaled, is an example. It is well known that there are infinitely many examples like this. This topic is reviewed in this paper, and many interesting properties of these eigenfunctions are discussed. Also discussed is a simple way to characterize all eigenfunctions of the Fourier transform.

1. INTRODUCTION

The Fourier transform of a signal $x(t)$, defined by the equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \quad (1)$$

has importance in all branches of science and engineering [2, 4]. Electrical engineers use the Fourier transform (FT) operator to analyze and design electrical circuits, linear systems, communication networks, and feedback control systems. The inverse of the FT is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad (2)$$

We say that $x(t)$ and $X(j\omega)$ form an FT pair and write $x(t) \Leftrightarrow X(j\omega)$. In this paper we focus on a very interesting aspect of the FT which is often not emphasized in the class room. This is the notion of eigenfunctions of the Fourier transform. If $x(t)$ is a nonzero function such that the Fourier transform has the same shape, that is,

$$X(j\omega) = \lambda x(\omega) \quad (3)$$

for constant λ, we say that $x(t)$ is an eigenfunction of the FT operator with eigenvalue λ. Thus, the shape is invariant to the FT operation. A commonly mentioned example is the Gaussian function, based on the following Fourier transform pair:

$$g(t) = e^{-t^2/2} \quad \Leftrightarrow \quad G(j\omega) = \sqrt{2\pi}e^{-\omega^2/2} \quad (4)$$

Thus, $g(t)$ is an eigenfunction with eigenvalue $\lambda = \sqrt{2\pi}$. The function $g(t)$, properly scaled, is the Gaussian probability density $e^{-t^2/2}/\sqrt{2\pi}$. For convenience we shall refer to the above pair (4) as the Gaussian FT pair.[2] Another example of an eigenfunction is the impulse train made from Dirac delta functions. This arises from the following Fourier transform pair [4]:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \quad \Leftrightarrow \quad S(j\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi n}{T}) \quad (5)$$

where $s(t)$ is the sampling function. For the specific spacing $T = \sqrt{2\pi}$, the function $s(t)$ becomes an eigenfunction, and the eigenvalue is again $\lambda = \sqrt{2\pi}$, as in the first example.

These examples bring up the following questions. (*a*) Are there more examples of eigenfunctions of the FT operator? (*b*) Are eigenvalues always equal to $\sqrt{2\pi}$?

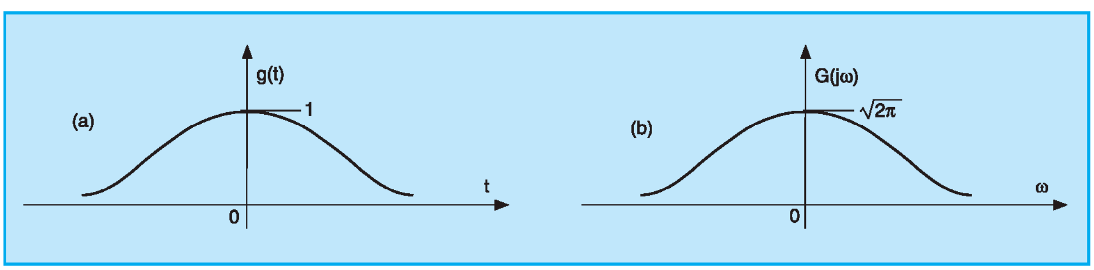

Fig 1 The Gaussian $g(t)$ and its Fourier transform $G(j\omega)$. These plots have exactly identical shape, so the Gaussian is an eigenfunction of the Fourier transform operator

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[2] Notice that a time-dilated version such as $x(t) = e^{-t^2/2\sigma^2}$ is not an eigenfunction because $X(j\omega) = \sqrt{2\pi\sigma^2}e^{-\omega^2\sigma^2/2}$. This $X(j\omega)$ has Gaussian shape all right, but it is dilated differently in frequency.

In this paper we shall answer these questions. In fact we shall go beyond that. We will give a complete list of all possible eigenvalues: there are only four! But then, there is an infinite number of eigenfunctions. We shall provide a complete characterization of all eigenfunctions. We will give a recipe capable of constructing every possible eigenfunction. The recipe is so simple, you can start from almost any Fourier transformable function and change it into an eigenfunction. Even though the eigenfunctions of the Fourier and related transforms have been discussed by a few authors (McClellan and Parks, 1972), (Soares, *et al.,* 2003), and (Pei and Ding, 2007), the presentation given here is simple and direct, and readily accessible to students with only an introductory background in signals and systems.

2. A METHOD TO CONSTRUCT EIGEN FUNCTIONS

First we point out a very simple trick to generate eigenfunctions. This will show that there is an unlimited number of these. From the inverse FT formula (2) we notice that

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(j\omega)e^{-j\omega t}d\omega \qquad (6)$$

Thus, if we take the Fourier transform of the Fourier transform we get the original function $x(t)$ modified in two ways: (a) it is time reversed, and (b) it is multiplied by the factor 2π. Consider a real function $x_e(t)$ and assume this is even, that is,

$$x_e(t) = x_e(-t) \qquad (7)$$

Then its transform $X_e(j\omega)$ is also real and even [4]:

$$X_e(j\omega) = X_e(-j\omega). \qquad (8)$$

Now consider the time domain function $X_e(jt)$, and take its FT. Then the result is $2\pi x_e(-\omega)$ (from (6)) which in turn equals $2\pi x_e(\omega)$ (by (7)). If we now define the sum

$$y_1(t) \triangleq x_e(t) + \frac{X_e(jt)}{\sqrt{2\pi}} \qquad (9)$$

then its Fourier transform is

$$\begin{aligned} Y_1(j\omega) &= X_e(j\omega) + \frac{2\pi x_e(\omega)}{\sqrt{2\pi}} \\ &= X_e(j\omega) + \sqrt{2\pi}x_e(\omega) \\ &= \sqrt{2\pi}\left(x_e(\omega) + \frac{X_e(j\omega)}{\sqrt{2\pi}}\right) = \sqrt{2\pi}y_1(\omega) \end{aligned}$$

Thus the Fourier transform $Y_1(j\omega)$ is exactly the original function $y_1(t)$ scaled by $\sqrt{2\pi}$. Summarizing, given any nonzero real even function $x_e(t)$, the function $y_1(t)$ defined as in (9) is an eigenfunction of the FT operator! In this example the eigenvalue is $\sqrt{2\pi}$ and the eigenfunction $y_1(t)$ is real and even, as in the Gaussian and impulse-train cases. Notice that given an arbitrary function $x(t)$, we can generate an eigenfunction from it by taking the even part of the real part and using it as $x_e(t)$ in the above construction.

2.1. Some specific examples

For a specific example, let $x_e(t)$ be the rectangular pulse $p(t)$ shown in Fig 2a, which is real and even. Its Fourier transform is the sinc function

$$P(j\omega) = \frac{\sin(\omega/2)}{\omega/2} \qquad (10)$$

shown in Fig 2b. The function

$$y_1(t) = p(t) + \frac{1}{\sqrt{2\pi}}\frac{\sin(t/2)}{t/2} \qquad (11)$$

shown in Fig 3 (top) is therefore an eigenfunction of the FT operator. Next let $x_e(t)$ be taken as the triangular pulse

$$q(t) = \begin{cases} 1-|t| & -1 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases} \qquad (12)$$

This is nothing but the convolution of the pulse $p(t)$ with itself. So the Fourier transform is

$$Q(j\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^2 \qquad (13)$$

The eigengfunction $y_1(t)$ constructed from $q(t)$ is therefore

$$y_1(t) = q(t) + \frac{1}{\sqrt{2\pi}}\left(\frac{\sin(t/2)}{t/2}\right)^2 \qquad (14)$$

This is also shown in Fig 3 (bottom). Notice that the first term in (11) is not continuous and the second term decays like $1/|t|$ for large $|t|$. In (14) the first term (triangle) is continuous and the second term decays faster, like $1/|t|^2$. If we convolve the pulse with itself one more time, the first term of (14) is replaced with a quadratic, which is not only continuous but also differentiable (even at the point where it becomes zero permanently). The corresponding second term is proportional to the cube of the sinc function. Repeating this idea, we can generate a whole family of real even eigenfunctions by replacing the first term with the "pulse convolved with itself N times". The second term then becomes the sinc function multiplied with itself N times. This is called a spline function (a B-spline to be more specific), and its properties have

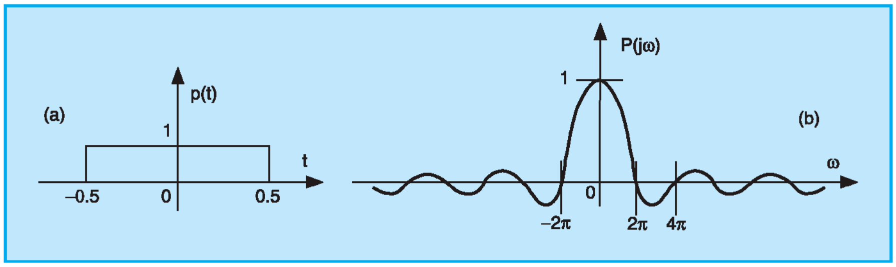

Fig 2 The rectangular pulse and its Fourier transform

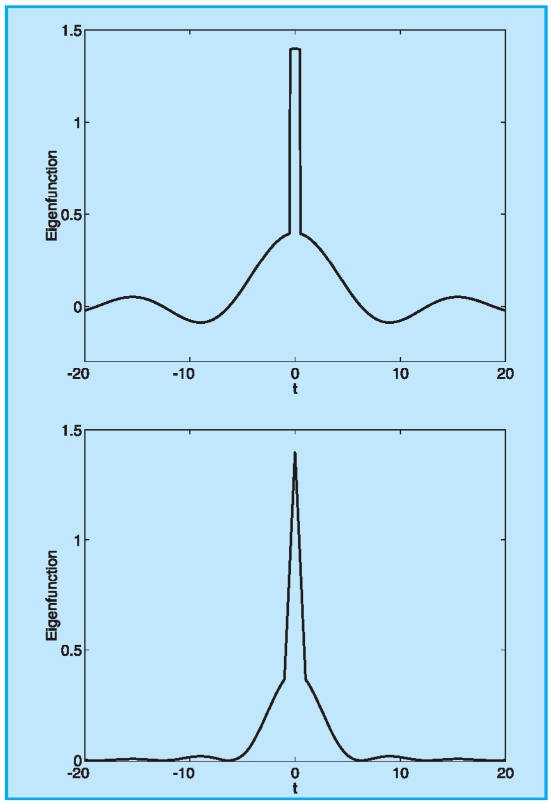

Fig 3 The eigenfuncation created from the rectangular pulse and its FT (top), and the eigenfunction created from the triangular pulse and its FT (bottom)

been studied extensively [9]. As N increases, the first term (the spline) becomes smoother and smoother (differentiable more and more number of times), and the second term decays faster and faster. Notice also that the second term is infinitely differentiable for any N. Thus we can generate a whole family of eigenfunctions from this example, which have specified smoothness and decay properties.

2.2. More general constructions

The preceding construction immediately reveals many other ways to get eigenfunctions of FT. For example if we replace (9) with

$$y_2(t) \triangleq x_e(t) - \frac{X_e(jt)}{\sqrt{2\pi}} \tag{15}$$

then we find that $Y_2(j\omega) = -\sqrt{2\pi} y_2(\omega)$. So $y_2(t)$ is a real-even eigenfunction, but the corresponding eigenvalue value is $-\sqrt{2\pi}$ instead of $\sqrt{2\pi}$. Next, let $x_o(t)$ be a real and odd function. Then the Fourier transform $X_o(j\omega)$ is imaginary and odd [4]. Define the real and odd function

$$y_3(t) \triangleq x_o(t) + \frac{X_o(jt)}{j\sqrt{2\pi}} \tag{16}$$

Then

$$
\begin{aligned}
Y_3(j\omega) &= X_o(j\omega) + \frac{2\pi x_o(-\omega)}{j\sqrt{2\pi}} \\
&= X_o(j\omega) - j\sqrt{2\pi}x_o(-\omega) \\
&= X_o(j\omega) + j\sqrt{2\pi}x_o(\omega) \\
&= j\sqrt{2\pi}\Big(x_o(\omega) + \frac{X_o(j\omega)}{j\sqrt{2\pi}}\Big) = j\sqrt{2\pi}y_3(\omega)
\end{aligned}
$$

proving that $y_3(t)$ is an eigenfunction with eigenvalue $j\sqrt{2\pi}$. Finally, if we change the example to

$$y_4(t) \triangleq x_o(t) - \frac{X_o(jt)}{j\sqrt{2\pi}} \tag{17}$$

which is also a real and odd function, we get

$$Y_4(j\omega) = -j\sqrt{2\pi}y_4(\omega) \tag{18}$$

That is, $y_4(t)$ is an eigenfunction with eigenvalue $-j\sqrt{2\pi}$. Summarizing, we have produced families of realeven eigenfunctions with eigenvalues $\pm\sqrt{2\pi}$, and families of real-odd eigenfunctions with eigenvalues $\pm j\sqrt{2\pi}$. The reader can create an unlimited number of examples by substituting any nonzero real-even function $x_e(t)$ into (9) or (15) or any nonzero real-odd function $x_o(t)$ into (16) or (17).

The next question is "Is this all there is to it, or are there more eigenfunctions out there?" It turns out that the only possible eigenvalues of the FT operator are indeed $\sqrt{2\pi}, -\sqrt{2\pi}, j\sqrt{2\pi}$, and $-j\sqrt{2\pi}$ as we shall show next. The set of all eigenfunctions is bigger than the four families of examples $y_1(t)$, $y_2(t)$, $y_3(t)$ and $y_4(t)$ given above. But it is only "slightly bigger" as we shall see (Sec. 4).

3. SET OF ALL EIGENVALUES

We now show formally that the Fourier transform operator can have only four possible eigenvalues. Let $x(t)$ be an eigenfunction of the FT. Then its FT is $\lambda x(\omega)$. Taking FT again we get $\lambda^2 x(\omega)$. If we take the Fourier transform four times on $x(t)$ we will get

$$\lambda^4 x(t). \tag{19}$$

But we know that taking FT twice always yields $2\pi x(-t)$ (see beginning of Sec. 2). So, taking FT four times must actually yield $(2\pi)^2 x(t)$ for any $x(t)$. So we necessarily have

$$\lambda^4 x(t) = (2\pi)^2 x(t) \tag{20}$$

This means that we necessarily have

$$\lambda^4 = (2\pi)^2 \tag{21}$$

This yields four possible choices for the eigenvalue, namely

$$\lambda = \sqrt{2\pi},\ -\sqrt{2\pi},\ j\sqrt{2\pi},\ -j\sqrt{2\pi} \tag{22}$$

It is clear from this derivation that these are the only possible eigenvalues of the FT operator. All of these possibilities have been demonstrated with the constructions shown in Sec. 2. Notice in particular that $\lambda^2 = +2\pi$.

4. COMPLETE FAMILY OF EIGENFUNCTIONS

We now make some general observations about eigenfunctions of the Fourier transform operator. This will also lead to a complete characterization.

4.1. *Complex eigenfunctions*

Even though all our examples so far have been real valued functions, complex examples are readily generated. Thus, if $x_1(t)$ and $x_2(t)$ are real eigenfunctions with the same eigenvalue $\sqrt{2\pi}$, then $x_1(t) + jx_2(t)$ has the FT

$$X_1(j\omega) + jX_2(j\omega) = \sqrt{2\pi}\Big(x_1(\omega) + jx_2(\omega)\Big) \tag{23}$$

Thus $x_1(t) + jx_2(t)$ is a complex eigenfunction with the same eigenvalue. The real and imaginary parts do not have to be related in any way. We now prove an important property of eigenfunctions:

Lemma 1. Let $x(t)$ be an eigenfunction (possibly complex) of the Fourier transform operator, with eigenvalue λ. Then the following are true:

1. $x(t)$ is necessarily an even or odd function. In particular, the real and imaginary parts are individually both even or both odd.
2. Both the real and imaginary parts are eigenfunctions with the same eigenvalue λ.

Proof. Given any eigenfunction $x(t)$, if we take the FT twice we get $\lambda^2 x(t)$. This should also be equal to $2\pi x(-t)$ (see beginning of Sec. 2). Since $\lambda^2 = \pm 2\pi$ (Sec. 3) we conclude

$$x(t) = \begin{cases} x(-t) & \text{if } \lambda = \pm\sqrt{2\pi} \\ -x(-t) & \text{if } \lambda = \pm j\sqrt{2\pi} \end{cases} \tag{24}$$

Thus $x(t)$ is even when $\lambda = +\sqrt{2\pi}$ and odd when $\lambda = \pm j\sqrt{2\pi}$. Writing $x(t) = x_1(t) + jx_2(t)$ this means that the real part $x_1(t)$ and imaginary part $x_2(t)$ are both even or both odd. Now look at

$$X(j\omega) = X_1(j\omega) + jX_2(j\omega) = \lambda\ (x_1(\omega) + jx_2(\omega)) \tag{25}$$

If $x_1(t)$ and $x_2(t)$ are both even then their Fourier transforms are real and even. Since $\lambda = +\sqrt{2\pi}$ from (24), eqn. (25) therefore implies

$$X_1(j\omega) = \lambda x_1(j\omega),\ X_2(j\omega) = \lambda x_2(j\omega), \tag{26}$$

That is, the real part and imaginary part are eigenfunctions with the same eigenvalue. The other possibility is that $x_1(t)$ and $x_2(t)$ are both odd. Then their Fourier transforms are imaginary and odd. Since $\lambda = \pm j\sqrt{2\pi}$ from (24), eqn (25) again implies (26). Again, the real part and imaginary part are both eigenfunctions, with the same eigenvalue.

Thus, writing the eigenfunction in the form

$$x(t) = x_1(t) + jx_2(t) \quad (27)$$

where $x_1(t)$ and $x_2(t)$ are real, there are precisely four possibilities as shown below:

λ (eigenvalue)	$x_1(t)$ and $x_2(t)$ (both real eigenfunctions)
$\sqrt{2\pi}$	even
$-\sqrt{2\pi}$	even
$j\sqrt{2\pi}$	odd
$-j\sqrt{2\pi}$	odd

Conversely, notice that if $x_1(t)$ and $x_2(t)$ fall under any of the four cases listed above, then it is clear that $x(t)$ is an eigenfunction with the same eigenvalue λ. We can summarize all these results as follows:

Theorem 1. The function $x(t)$ is an eigenfunction of the Fourier transform operator with eigenvalue λ if and only if its real and imaginary parts $x_1(t)$ and $x_2(t)$ are both eigenfunctions with eigenvalue λ, and satisfy one of the four cases: either they are both even with common eigenvalue $\sqrt{2\pi}$ or $-\sqrt{2\pi}$, or they are both odd with common eigenvalue $j\sqrt{2\pi}$ or $-j\sqrt{2\pi}$.

4.2. *Generating all eigenfunctions*

From Theorem 1 we see that the set of all eigenfunctions can be "generated" by generating all real-even eigenfunctions and real-odd eigenfunctions. We conclude by showing that any real even eigenfunction has the form

$$y_1(t) \triangleq x_e(t) + \frac{X_e(jt)}{\sqrt{2\pi}}, \quad \text{or} \quad y_2(t) \triangleq x_e(t) - \frac{X_e(jt)}{\sqrt{2\pi}} \quad (28)$$

where $x_e(t)$ is a real and even function with Fourier transform $X_e(j\omega)$ (which is also real and even). Similarly any real odd eigenfunction has the form

$$y_3(t) \triangleq x_o(t) + \frac{X_o(jt)}{j\sqrt{2\pi}} \quad \text{or} \quad y_4(t) \triangleq x_o(t) - \frac{X_o(jt)}{j\sqrt{2\pi}} \quad (29)$$

where $x_o(t)$ is a real and odd function with Fourier transform $X_o(j\omega)$ (which is imaginary and odd).

Proof: First let $y(t)$ be a real and even eigenfunction. Since $Y(j\omega)$ is also real and even, $Y(j\omega) = +$ $Y(j\omega) = \pm\sqrt{2\pi}y(\omega)$. So we can write, rather trivially,

$$y(t) = \frac{y(t)}{2} \pm \frac{Y(jt)}{2\sqrt{2\pi}} \quad (30)$$

Identifying $x_e(t) = 0.5y(t)$ we see that $y(t)$ indeed has one of the forms (28). Next let $y(t)$ be a real and odd eigenfunction. Since $Y(j\omega)$ is imaginary and odd, we have $Y(j\omega) = \pm j\sqrt{2\pi}y(\omega)$. So we can write

$$y(t) = \frac{y(t)}{2} \pm \frac{Y(jt)}{2j\sqrt{2\pi}} \quad (31)$$

Identifying $x_o(t) = 0.5y(t)$ we see that $y(t)$ has one of the forms (29).

Since we have already shown (Sec. 2) that (28) and (29) are eigenfunctions for any real even $x_e(t)$ and real odd $x_o(t)$, this proves the following:

Theorem 2. A real function is an eigenfunction of the Fourier transform operator if and only if it has one of the four forms $yk(t)$ in (28) and (29), where $x_e(t)$ is a real even function (with FT denoted as $X_e(j\omega)$) and $x_o(t)$ is a real odd function (with FT denoted as $Xo(j\omega)$). The corresponding eigenvalue is (a) $\sqrt{2\pi}$ for $y_1(t)$, (b) $-\sqrt{2\pi}$ for $y_2(t)$, (c) $j\sqrt{2\pi}$ for $y_3(t)$, and (d) $-j\sqrt{2\pi}$ for $y_4(t)$.

Summary. Given an arbitrary real-even function $x_e(t)$ or real-odd function $x_o(t)$, define a real function $y_k(t)$ as in eqn (28) or (29). Then assuming it is nonzero, $yk(t)$ is an eigenfunction of the FT operator, with eigenvalue $\pm\sqrt{2\pi}$ or $\pm j\sqrt{2\pi}$ as summarized in Theorem 2. Furthermore, all real eigenfunctions can be generated this way. By using two real eigenfunctions $x_1(t)$ and $x_2(t)$ with the same eigenvalue (i.e., of the same type out of the four types in eqn (28) and (29)), we can generate a complex eigenfuncton $x_1(t)+jx_2(t)$ with the same eigenvalue. Moreover, *all* complex eigenfunctions can be generated this way.

5. OTHER TRICKS TO GENERATE EIGENFUNCTIONS

Even though the method described in the preceding section can generate any eigenfunction of the FT operator, it is useful to know other methods which have been around in the literature. Here we describe a clever method based on derivatives. The method is based on the following simple observation. Let $x(t) \Leftrightarrow X(j\omega)$ be a Fourier transform pair. Then the Fourier transform of $dx(t)/dt$ is $j\omega X(j\omega)$ whereas the Fourier transform of $tx(t)$ is $jdX(j\omega)/d\omega$ [4]. Thus we have the FT pairs

$$\frac{dx(t)}{dt} \Leftrightarrow j\omega X(j\omega) \quad (32)$$

and

$$tx(t) \Leftrightarrow j\frac{dX(j\omega)}{d\omega} \quad (33)$$

Subtraction yields the FT pair:

$$tx(t) - \frac{dx(t)}{dt} \Leftrightarrow -j\Big(\omega X(j\omega) - \frac{dX(j\omega)}{d\omega}\Big) \quad (34)$$

Now assume that we have a differentiable eigenfunction $f_0(t)$ of FT, so that its FT is $\lambda f_0(\omega)$. Substituting $x(t) = f_0(t)$ and $X(j\omega) = \lambda f_0(\omega)$ into (34) we obtain the FT pair:

$$tf_0(t) - \frac{df_0(t)}{dt} \Leftrightarrow -j\lambda\Big(\omega f_0(\omega) - \frac{df_0(\omega)}{d\omega}\Big) \quad (35)$$

That is, whenever $f_0(t)$ is an eigenfunction with eigenvalue λ, the signal defined as

$$f_1(t) \triangleq tf_0(t) - \frac{df_0(t)}{dt} \quad (36)$$

is also an eigenfunction, and has eigenvalue $-j\lambda$. If $f_0(t)$ is twice differentiable, we can repeat this process, that is, we can generate a function

$$f_2(t) \triangleq tf_1(t) - \frac{df_1(t)}{dt} \quad (37)$$

This is an eigenfunction with eigenvalue $-j(-j\lambda) = -\lambda$. Repeated application of this recursive process yields eigenfunctions $f_k(t)$ with eigenvalues

$$\ldots \sqrt{2\pi}, -j\sqrt{2\pi}, -\sqrt{2\pi}, j\sqrt{2\pi}, \ldots \quad (38)$$

This pattern of eigenvalues repeats periodically. For a specific example let $f_0(t)$ be the Gaussian eigenfunction $f_0(t) = e^{-t^2/2}$ (Sec. 1). Then

$$f_1(t) = 2te^{-t^2/2} \quad (39)$$

Since $f_0(t)$ has Fourier transform $\sqrt{2\pi}f_0(\omega)$, the function $f_1(t)$ has Fourier transform $-j\sqrt{2\pi}f_1(\omega)$. This establishes the Fourier transform pair

$$2te^{-t^2/2} \Leftrightarrow -2j\sqrt{2\pi}\,\omega e^{-\omega^2/2} \quad (40)$$

Next,

$$f_2(t) = tf_1(t) - \frac{df_1(t)}{dt} = (4t^2-2)e^{-t^2/2} \quad (41)$$

Since $f_1(t)$ has the Fourier transform $-j\sqrt{2\pi}f_1(\omega)$, we see that $f_2(t)$ has the Fourier transform

$$-j[-j\sqrt{2\pi}f_2(\omega)] = -\sqrt{2\pi}f_2(\omega) = -\sqrt{2\pi}(4\omega^2-2)e^{-\omega^2/2} \quad (42)$$

Thus $f_2(t)$ yields the Fourier transform pair

$$(4t^2-2)e^{-t^2/2} \Leftrightarrow -\sqrt{2\pi}(4\omega^2-2)e^{-\omega^2/2}. \quad (43)$$

This process can be continued indefinitely. Figure 4 shows plots of the first three eigenfunctions $f_0(t)$, $f_1(t)$, and $f_2(t)$ in this series. Notice that $f_k(t)$ is an odd eigenfunction for odd k and even eigenfunction for even k. The eigenfunctions presented above have the form

$$f_k(t) = H_k(t)e^{-t^2}/2 \quad (44)$$

where $H_k(t)$ are polynomials:

$$H_0(t) = 1,\ H_1(t) = 2t,\ H_2(t) = 4t^2 - 2,\ldots \quad (45)$$

and so on. It can be veri.ed that these polynomials can be expressed in the form

$$H_k(t) = (-1)^k e^{t^2}\Big(\frac{d^k}{dt^k}e^{-t^2}\Big) \quad (46)$$

These are called the Hermite polynomials. The family of eigenfunctions described above arise naturally in problems involving Schrodinger's equation in quantum mechanics (Davies, 1984).

As a final remark notice that instead of subtraction in (34) one could add:

$$tx(t) + \frac{dx(t)}{dt} \Leftrightarrow j\Big(\omega X(j\omega) + \frac{dX(j\omega)}{d\omega}\Big) \quad (47)$$

So if $x(t)$ is an eigenfunction, then the left hand side is also one, as long as it is nonzero. We can start from any di.erentiable eigenfunction $f_0(t)$ and generate a whole series like this. If we start from the Gaussian $f_0(t) = e^{-t^2/2}$ then this particular construction is not very exciting because $f_1(t) = 0$, and so all $f_k(t)$, $k > 1$, are zero!

6. CONCLUDING REMARKS

In this paper we used ω as the frequency variable. If we let $\omega = 2\pi f$ and define the Fourier transform to be

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt, \quad (48)$$

then the inverse of the FT is given by

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \quad (49)$$

This convention is often used in communication literature. It has the advantage that the transform and its inverse look symmteric (without the 2π factor discrepancy). The only difference between (48) and (49) is the negative sign in the exponent. With the FT operator defined as the transformation from $x(t)$ to $X(f)$, the four eigenvalues become

$$1,\ -1,\ j,\ -j \quad (50)$$

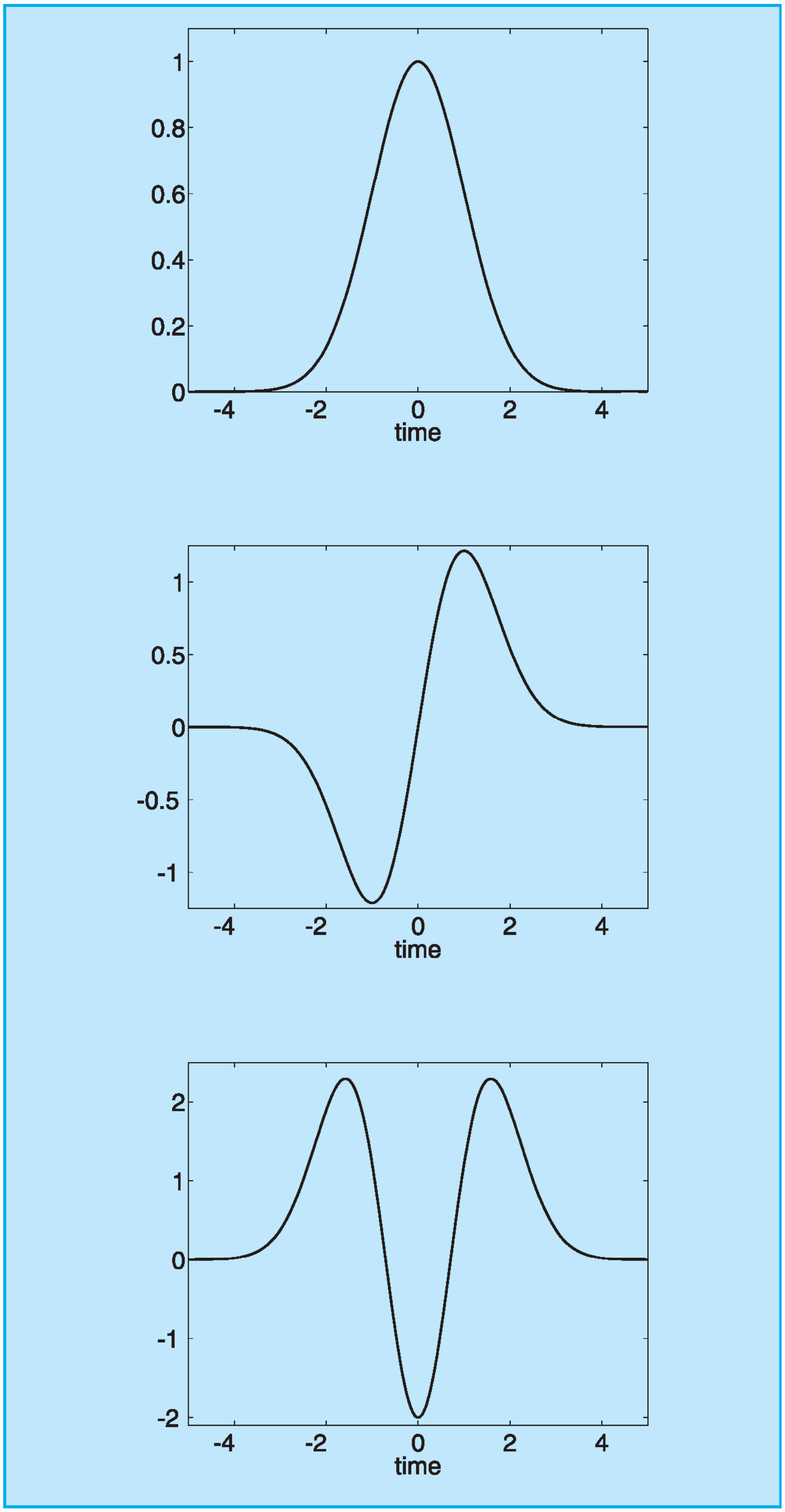

Fig 4 Plots of some of the eigenfunctions derived from the Gaussian. See text. From top to bottom: $f_0(t)$, $f_1(t)$, and $f_2(t)$

That is, the $\sqrt{2\pi}$ factor is not there. The use of f therefore makes the results more beautiful. For example, the impulse-train FT pair (5) becomes, with $T = 1$,

$$\sum_{n=-\infty}^{\infty} \delta(t-n) \Leftrightarrow \sum_{k=-\infty}^{\infty} \delta(f-k) \tag{51}$$

We have used ω in this paper because it appears to be more commonly used in introductory classes. It is interesting here to recall that an $N \times N$ unitary matrix always has eigenvalues of the form $e^{j\theta}$ where θ is real. The eigenvalues (50) of the FT operator are therefore a restricted class of eigenvalues of unitary operators.

Another remark relates to the discrete Fourier transform or the DFT. This converts a set of N numbers $x(n)$ into another set of N numbers $X[k]$ using the formula

$$X[k] = \sum_{n=0}^{N-1} x(n) W^{nk} \tag{52}$$

where $W = e^{-j2\pi/N}$. For a finite duration sequence $x(n)$, the DFT is nothing but the set of samples of the Fourier transform at N uniformly spaced frequencies in $0 < \omega < 2\pi$. The DFT operator is nothing but a matrix-vector multiplication (involving a unitary matrix) converting a vector of N components into another. An interesting question is, what are the eigenvalues and eigenvectors of the DFT operator? This question was addressed by McClellan and Parks (1972) who showed that out of the set of N eigenvalues, there are only four distinct eigenvalues. These are given by

$$\sqrt{N},\ -\sqrt{N},\ j\sqrt{N},\ -j\sqrt{N} \tag{53}$$

The characterization of the eigenvectors is more complicated, as described in great detail in (McClellan and Parks, 1972). Eigenfunctions of many other operators related to the Fourier transform have been discussed by some authors. The interested student should read (Pei and Ding, 2007) and references therein.

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