RESULTS ON VECTOR BIORTHOGONAL PARTNERS

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ABSTRACT

The concept of Multiple Input Multiple Output (MIMO) biorthogonal partners arises in many different contexts, one of them being multiwavelet theory. They also play a central role in the theory of MIMO channel equalization, especially with fractionally spaced equalizers. In this paper we will explore some further theoretical properties of MIMO biorthogonal partners. These include the conditions for the existence of MIMO biorthogonal partners and their application in finding the solution for the least squares signal approximation problem.[1]

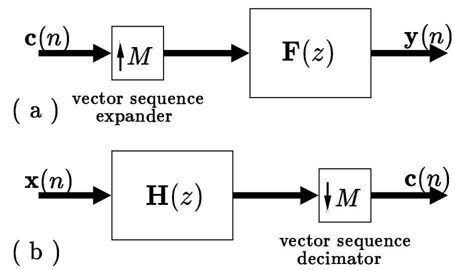

Figure 1: Least squares signal modelling: (a) signal model and (b) least squares solution (see text).

1. INTRODUCTION

Digital filters $H(z)$ and $F(z)$ are called **biorthogonal partners** of each other with respect to an integer M if their cascade $H(z)F(z)$ obeys the Nyquist(M) property [2]. In the multiple input multiple output (MIMO) case, biorthogonal partners are defined using a similar approach [1]. However, in this case the "biorthogonal partner" relation is not symmetric, so we distinguish between a *left biorthogonal partner* (**LBP**) and a *right biorthogonal partner* (**RBP**). Before introducing the new results, we will give a brief overview of several different contexts in which MIMO biorthogonal partners occur.

1.1. Motivation

Suppose we are given the signal model as shown in Fig. 1(a). The vector signal $\mathbf{y}(n)$ is obtained by upsampling the vector sequence $\mathbf{c}(n)$ and passing the result through the matrix transfer function $\mathbf{F}(z)$. Now, given a vector signal $\mathbf{x}(n)$, suppose we want to approximate it by a signal $\mathbf{y}(n)$ admitting the described model. The optimum vector sequence $\mathbf{c}(n)$ is then determined as in Fig. 1(b). The prefilter $\mathbf{H}(z)$ turns out to be a particular form of a MIMO biorthogonal partner of $\mathbf{F}(z)$. In the following we refer to this as the least squares problem. A very similar problem arises in multiwavelet theory [8]. Consider the two-band multiwavelet transform. The space V_0 is spanned by N scaling functions and their integer shifts. Similarly, the space W_0 is spanned by N wavelets and their integer shifts. Those two spaces together form a finer resolution space V_1. Suppose we have a signal $x_1(n)$ belonging to the space V_1 and we want to find a coarser signal $x_0(n)$ from V_0 such that the distance (in the ℓ_2 sense) from the signal $x_1(n)$ is minimized. This problem can be formulated as a vector valued least squares problem, so the solution is again given by Fig. 1.

Another place where MIMO biorthogonal partners occur is the equalization of vector channels. Figure 2 shows a MIMO communication channel employing the fractionally spaced equalizer at the receiver. It was shown in [1] that the FSE needs to be a LBP of the equivalent channel transfer matrix. Moreover, an algorithm was proposed that exploits the flexibility in the design of LBP, so that the system in Fig. 2 becomes more robust to the channel noise.

In this paper we will explore some theoretical properties of MIMO biorthogonal partners that were not considered in [1]. We first provide the definition of a MIMO biorthogonal partner. Then we give a necessary and sufficient condition for the existence of (a stable) MIMO biorthogonal partner. Finally we consider the least squares problem and provide the solution within the MIMO biorthogonal partner setting.

1.2. Notations

If not stated otherwise, all notations are as in [3]. We use the notation $[x(n)]_{\downarrow M}$ and $[X(z)]_{\downarrow M}$ to denote the decimated

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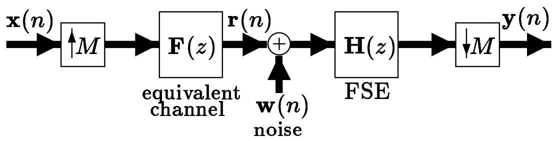

Figure 2: Discrete-time equivalent communication channel with FSE.

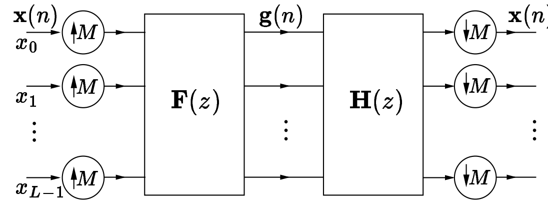

Figure 3: Block diagram interpretation of a left biorthogonal partner.

version $x(Mn)$ and its z-transform. The expanded version

$$\begin{cases} x(n/M) & \text{for } n = \text{mul of } M, \\ 0 & \text{otherwise} \end{cases}$$

is indicated by $[x(n)]_{\uparrow M}$, and its z-transform $X(z^M)$ is denoted by $[X(z)]_{\uparrow M}$. In a block diagram, the decimation and expansion operations are represented by symbols $\downarrow M$ and $\uparrow M$ respectively. In the case of vectors signals (e.g. Fig. 1 and Fig. 2), the decimation and expansion are performed on each element separately. The polyphase decompozition [3] is also valid in the matrix case. Thus for example if $\mathbf{F}(z)$ is a matrix transfer function, then it can be written in the Type-2 polyphase form as

$$\mathbf{F}(z) = \sum_{k=0}^{M-1} z^k \mathbf{F}_k(z^M). \tag{1}$$

If not mentioned otherwise, all the matrices in this paper are rectangular. It is implicit that their dimensions are such that the matrix products in question are well defined and that the product matrices have the appropriate size.

2. REVIEW OF MIMO BIORTHOGONAL PARTNERS

In this section by review the notion of a MIMO biorthogonal partner and introduce its most general form (see also [1]).

Definition 1. MIMO Biorthogonal partners. A MIMO transfer function $\mathbf{H}(z)$ is said to be a left biorthogonal partner (LBP) of $\mathbf{F}(z)$ with respect to an integer M if

$$[\mathbf{H}(z)\mathbf{F}(z)]_{\downarrow M} = \mathbf{I}. \tag{2}$$

Similarly, a MIMO transfer function $\mathbf{H}(z)$ is said to be a right biorthogonal partner (RBP) of $\mathbf{F}(z)$ with respect to an integer M if $[\mathbf{F}(z)\mathbf{H}(z)]_{\downarrow M} = \mathbf{I}$.

The interpretation of the first part of the above definition is shown in Fig. 3. It can be seen that if $\mathbf{H}(z)$ is a LBP of $\mathbf{F}(z)$, it implies that $\mathbf{F}(z)$ is a RBP of $\mathbf{H}(z)$, but it does not imply that $\mathbf{H}(z)$ is also a RBP of $\mathbf{F}(z)$. However, as pointed out in [1], the results that hold for LBPs can easily be modified to hold for RBPs. That is why we will only consider left biorthogonal partners in the following. The other important point to make here is that if M is changed, the two filters might not remain partners. However, we will often omit the term "with respect to M", since it will usually be understood from the context.

As it was shown in [1], it is possible to state the most general form of biorthogonal partners. For example, we can say that a MIMO transfer function $\mathbf{H}(z)$ is a LBP of $\mathbf{F}(z)$ if and only if it can be expressed in the form

$$\mathbf{H}(z) = ([\mathbf{G}(z)\mathbf{F}(z)]_{\downarrow M \uparrow M})^{-1}\mathbf{G}(z) \tag{3}$$

for some MIMO transfer function $\mathbf{G}(z)$ of the same size as $\mathbf{H}(z)$. Similarly, $\mathbf{H}(z)$ is a RBP of $\mathbf{F}(z)$ if and only if it can be expressed in the form

$$\mathbf{H}(z) = \mathbf{G}(z)([\mathbf{F}(z)\mathbf{G}(z)]_{\downarrow M \uparrow M})^{-1} \tag{4}$$

for the appropriate MIMO transfer function $\mathbf{G}(z)$.

3. THE MAIN RESULTS

In the following we present a necessary and sufficient condition on a MIMO transfer function $\mathbf{F}(z)$ for the existence of its MIMO biorthogonal partner $\mathbf{H}(z)$. From now on, by "existence of a biorthogonal partner" we actually mean "existence of a stable biorthogonal partner". The theorem and the corresponding corollary will be stated only for the case of left biorthogonal partners, but as mentioned earlier, by simple "transposition" they can be modified to hold for RBPs as well. After that we state the vector valued least squares problem in the general form. Theorem 2 explains the algorithm for solving this problem and the corresponding corollary deals with the uniqueness of the proposed solution. The proofs of those results can be found in Sec. 4.

Theorem 1. Existence of LBP. A MIMO transfer function $\mathbf{F}(z)$ with the Type-2 polyphase form as in (1) has a LBP if and only if for all ω in $[0, 2\pi)$ the following holds: if for all k ($0 \leq k \leq M-1$) we have $\mathbf{F}_k(e^{j\omega})\mathbf{C}(e^{j\omega}) = \mathbf{0}$, for some common vector $\mathbf{C}(e^{j\omega})$, then $\mathbf{C}(e^{j\omega}) \equiv \mathbf{0}$.

Therefore, for any fixed ω there cannot exist a nonzero common annihilating vector $\mathbf{C}(e^{j\omega})$ for all the M polyphase components of $\mathbf{F}(e^{j\omega})$. Note that in order for $\mathbf{F}(z)$ to have an *inverse* we need to have $\det[\mathbf{F}(e^{j\omega})] \neq 0$, for all ω, and this condition is stricter than the one in Theorem 1.

Unfortunately, the statement of Theorem 1 does not provide much intuition about the existence of biorthogonal partners. It turns out that if $\mathbf{F}(z)$ has any LBP, the choice (10) will be a valid one. This is a straightforward consequence of the following corollary, which is stated without proof since it follows directly from the proof of Theorem 1.

Corollary 1. A MIMO transfer function $\mathbf{F}(z)$ has a LBP if and only if $\mathbf{S}(\omega) = [\mathbf{F}^\dagger(e^{j\omega})\mathbf{F}(e^{j\omega})]_{\downarrow M}$ is a positive definite matrix for all ω in the range $[0, 2\pi)$.

Next, we consider the least squares problem as motivated in the introduction. This topic has been treated extensively in the setting of *oblique projections* [6]. In the scalar case, a similar problem is very common in multiresolution theory [4] as well as spline approximation theory [5], [2]. See also [7] for a slightly different problem formulation.

Consider the space $\mathcal{F}$ of all signals $\mathbf{y}(n)$ such that

$$\mathbf{Y}(z) = \mathbf{F}(z)\mathbf{C}(z^M) \tag{5}$$

where $\mathbf{c}(n)$ is an arbitrary ℓ_2 vector sequence.[2] This situation is depicted in Fig. 1(a). Here $\mathbf{F}(z)$ is a given MIMO transfer function. The problem is as follows. Given any vector signal $\mathbf{x}(n)$, we want to find the corresponding **projection** in $\mathcal{F}$, i.e. a vector signal $\mathbf{y}(n) \in \mathcal{F}$ such that

$$\sum_n \|\mathbf{y}(n) - \mathbf{x}(n)\|^2 \tag{6}$$

is minimized. Here $\|\cdot\|$ denotes the vector norm in ℓ_2. The following theorem describes the algorithm by which this is achieved and the corresponding corollary will address the uniqueness of the proposed solution.

Theorem 2. Solution to least squares problem. Given a MIMO transfer function $\mathbf{F}(z)$ and assuming that $\mathbf{S}(e^{j\omega}) = [\mathbf{F}^\dagger(e^{j\omega})\mathbf{F}(e^{j\omega})]_{\downarrow M}$ is a positive definite matrix for all ω, we define the (orthogonal) **projection filter** by

$$\mathbf{H}(z) = \left([\tilde{\mathbf{F}}(z)\mathbf{F}(z)_{\downarrow M}\right)^{-1}_{\uparrow M} \tilde{\mathbf{F}}(z). \tag{7}$$

If we pass the vector signal $\mathbf{x}(n)$ through the projection filter and decimate the outputs by M we get the optimal driving sequence $\mathbf{c}(n)$ (see Fig. 1(b)). This $\mathbf{c}(n)$ can be used to find the least squares approximation $\mathbf{y}(n)$ as in Fig. 1(a).

The positive-definiteness condition in Theorem 2 is necessary only to ensure the stability of $\mathbf{H}(z)$. The next corollary states that the least squares solution proposed by Theorem 2 is unique. The proof of Corollary 2 is omitted, since it closely follows the corresponding proof in the scalar case [2] and is also a direct consequence of the uniqueness of the orthogonal projection onto a closed subspace [6].

Corollary 2. Uniqueness of projection filter. Consider Fig. 1. For fixed $\mathbf{F}(z)$ satisfying the condition of Theorem 2 and $\mathbf{x}(n) \in \ell_2$, the least squares approximation $\mathbf{y}(n)$ is unique. Next, suppose the prefilter $\mathbf{H}(z)$ in Fig. 1(b) is such that the output of $\mathbf{F}(z)$ (Fig. 1(a)) is the least squares approximation of $\mathbf{x}(n)$ for any choice of the ℓ_2 input $\mathbf{x}(n)$. Then $\mathbf{H}(z)$ is unique and is therefore given by (7).

[2] This means that all the scalar sequences corresponding to the vector entries are square summable.

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. We start by proving the forward part of the theorem, i.e. supposing $\mathbf{H}(z)$ is a stable LBP of $\mathbf{F}(z)$, we need to show that there cannot exist a nonzero common annihilating vector $\mathbf{C}(e^{j\omega})$. By the supposition we have that $[\mathbf{H}(z)\mathbf{F}(z)]_{\downarrow M} = \mathbf{I}$, and this implies that there cannot exist a nonzero vector $\mathbf{C}(z)$ such that $\mathbf{F}(z)\mathbf{C}(z^M) = \mathbf{0}$. Indeed, if we assume there exists such nonzero vector $\mathbf{C}(z)$, we end up with the following contradiction

$$\mathbf{0} = [\mathbf{H}(z)\mathbf{F}(z)\mathbf{C}(z^M)]_{\downarrow M} = \mathbf{C}(z).$$

Rewriting $\mathbf{F}(z)$ in the Type-2 polyphase form (1) we then have that there cannot exist a nonzero vector $\mathbf{C}(z)$ such that

$$\sum_{k=0}^{M-1} z^k \mathbf{F}_k(z^M)\mathbf{C}(z^M) = \mathbf{0}$$

or equivalently, such that

$$\mathbf{F}_k(z)\mathbf{C}(z) = \mathbf{0} \quad \forall k, \;\; 0 \le k \le M-1.$$

Therefore, if there exists a stable LBP of $\mathbf{F}(z)$, then there cannot exist a common nonzero annihilating vector $\mathbf{C}(e^{j\omega})$ for all the M polyphase components $\mathbf{F}_k(e^{j\omega})$.

Now we proceed to prove the converse. For that, we suppose that for no ω does there exist a common nonzero vector $\mathbf{C}(e^{j\omega})$ annihilating $\mathbf{F}_k(e^{j\omega})$ for all k. This implies that the following matrix $\mathbf{S}(\omega)$ is positive definite for all ω

$$\mathbf{S}(\omega) = \sum_{k=0}^{M-1} \mathbf{F}_k^\dagger(e^{j\omega})\mathbf{F}_k(e^{j\omega}). \tag{8}$$

To justify this, recall that for any nonzero vector $\mathbf{C}(e^{j\omega})$ and $\mathbf{S}(\omega)$ as in (8) the entity $\mathbf{C}^\dagger(e^{j\omega})\mathbf{S}(\omega)\mathbf{C}(e^{j\omega})$ is a summation of nonnegative terms. Moreover, as asserted previously, for any choice of $\mathbf{C}(e^{j\omega})$ at least one of those terms is strictly positive, so that the overall result is positive. Observe from (8) that $\mathbf{S}(\omega) = [\mathbf{F}^\dagger(e^{j\omega})\mathbf{F}(e^{j\omega})]_{\downarrow M}$. Therefore, by the previous discussion we have

$$\det\left([\mathbf{F}^\dagger(e^{j\omega})\mathbf{F}(e^{j\omega})]_{\downarrow M}\right) > 0. \tag{9}$$

The final conclusion is that if there does not exist a common nonzero annihilating vector $\mathbf{C}(e^{j\omega})$ for all the M polyphase components $\mathbf{F}_k(e^{j\omega})$ then $\mathbf{F}(z)$ has a stable LBP. In particular, one such LBP is given by

$$\mathbf{H}(z) = \left([\tilde{\mathbf{F}}(z)\mathbf{F}(z)_{\downarrow M}\right)^{-1}_{\uparrow M} \tilde{\mathbf{F}}(z) \tag{10}$$

and is obtained from (3), with $\mathbf{G}(z) = \tilde{\mathbf{F}}(z)$. This LBP is stable due to (9), which concludes the proof. $\bigtriangledown\bigtriangledown\bigtriangledown$

Proof of Theorem 2. The error (6) that needs to be minimized can be rewritten in the frequency domain

$$\sum_n \|\mathbf{y}(n) - \mathbf{x}(n)\|^2 = \int_0^{2\pi} \|\mathbf{Y}(e^{j\omega}) - \mathbf{X}(e^{j\omega})\|^2 \frac{d\omega}{2\pi}$$
$$= \int_0^{2\pi} \underbrace{\|\mathbf{F}(e^{j\omega})\mathbf{C}(e^{j\omega M}) - \mathbf{X}(e^{j\omega})\|^2}_{\mathcal{E}(\omega)} \frac{d\omega}{2\pi}.$$

Note that $\mathbf{C}(e^{j\omega M})$ appearing in the integrand is periodic with period $2\pi/M$, and therefore can be chosen independently only in the range $0 \le \omega \le 2\pi/M$. That is why the integrand can be rewritten as

$$\mathcal{E}(\omega) = \sum_{k=0}^{M-1} \|\mathbf{F}(e^{j(\omega+\frac{2\pi k}{M})})\mathbf{C}(e^{j\omega M}) - \mathbf{X}(e^{j(\omega+\frac{2\pi k}{M})})\|^2.$$

For each ω in $0 \le \omega \le 2\pi/M$ we can choose $\mathbf{C}(e^{j\omega M})$ such that the nonnegative integrand $\mathcal{E}(\omega)$ is minimized and that would in turn minimize the projection error (6). Define the vector $\mathbf{a}(\omega)$ and the matrix $\mathbf{B}(\omega)$ as

$$\mathbf{a}(\omega) = [\mathbf{X}^T(e^{j\omega}) \;\; \mathbf{X}^T(e^{j(\omega+\frac{2\pi}{M})}) \cdots \mathbf{X}^T(e^{j(\omega+\frac{2\pi(M-1)}{M})})]^T$$
$$\mathbf{B}(\omega) = [\mathbf{F}^T(e^{j\omega}) \;\; \mathbf{F}^T(e^{j(\omega+\frac{2\pi}{M})}) \cdots \mathbf{F}^T(e^{j(\omega+\frac{2\pi(M-1)}{M})})]^T$$

The problem now reduces to that of minimizing

$$\begin{aligned}\mathcal{E}(\omega) &= \|\mathbf{B}(\omega)\mathbf{C}(e^{j\omega M}) - \mathbf{a}(\omega)\|^2 \\ &= [\mathbf{C}^\dagger(e^{j\omega M}) - \mathbf{a}^\dagger(\omega)\mathbf{B}(\omega)\mathbf{S}^{-1}(\omega)]\mathbf{S}(\omega)\cdot \\ &\quad \cdot[\mathbf{C}(e^{j\omega M}) - \mathbf{S}^{-1}(\omega)\mathbf{B}^\dagger(\omega)\mathbf{a}(\omega)] + \mathbf{a}^\dagger(\omega)\mathbf{a}(\omega) \\ &\quad -\mathbf{a}^\dagger(\omega)\mathbf{B}(\omega)\mathbf{S}^{-1}(\omega)\mathbf{B}^\dagger(\omega)\mathbf{a}(\omega) \qquad (11)\end{aligned}$$

where $\mathbf{S}(\omega) = \mathbf{B}^\dagger(\omega)\mathbf{B}(\omega)$. The form (11) was obtained by the "completion of squares". Consider the right hand side of the last equality in (11). It consists of two parts; the first part depends on the choice of $\mathbf{C}(e^{j\omega M})$ and the second part does not. Since the first part is always nonnegative, we should choose $\mathbf{C}(e^{j\omega M})$ such that it becomes zero. Note that the matrix $\mathbf{S}(\omega) = \mathbf{B}^\dagger(\omega)\mathbf{B}(\omega)$ is positive definite, which follows from the assumption $[\mathbf{F}^\dagger(e^{j\omega})\mathbf{F}(e^{j\omega})]_{\downarrow M} > 0$. Therefore, the only way to make the first part zero is to choose $\mathbf{C}(e^{j\omega M}) = \left(\mathbf{B}^\dagger(\omega)\mathbf{B}(\omega)\right)^{-1}\mathbf{B}^\dagger(\omega)\mathbf{a}(\omega)$. In order to rewrite this solution in terms of multirate building blocks, we note [3] that for any transfer function $\mathbf{A}(e^{j\omega})$, $[\mathbf{A}(e^{j\omega})]_{\downarrow M} = \frac{1}{M}\sum_{k=0}^{M-1}\mathbf{A}(e^{j\frac{\omega+2\pi k}{M}})$. Therefore,

$$\begin{aligned}\mathbf{B}^\dagger(\omega)\mathbf{B}(\omega) &= \sum_{k=0}^{M-1}\mathbf{F}^\dagger(e^{j(\omega+\frac{2\pi k}{M})})\mathbf{F}(e^{j(\omega+\frac{2\pi k}{M})}) \\ &= M[\mathbf{F}^\dagger(e^{j\omega})\mathbf{F}(e^{j\omega})]_{\downarrow M\uparrow M},\end{aligned}$$

$$\begin{aligned}\mathbf{B}^\dagger(\omega)\mathbf{a}(\omega) &= \sum_{k=0}^{M-1}\mathbf{F}^\dagger(e^{j(\omega+\frac{2\pi k}{M})})\mathbf{X}(e^{j(\omega+\frac{2\pi k}{M})}) \\ &= M[\mathbf{F}^\dagger(e^{j\omega})\mathbf{X}(e^{j\omega})]_{\downarrow M\uparrow M}.\end{aligned}$$

The optimal $\mathbf{C}(e^{j\omega M})$ is therefore

$$\mathbf{C}(e^{j\omega M}) = \Big[\left([\mathbf{F}^\dagger(e^{j\omega})\mathbf{F}(e^{j\omega})]_{\downarrow M}\right)^{-1}_{\uparrow M}\mathbf{F}^\dagger(e^{j\omega})\mathbf{X}(e^{j\omega})\Big]_{\downarrow M\uparrow M}.$$

Thus we have $\mathbf{C}(z) = [\mathbf{H}(z)\mathbf{X}(z)]_{\downarrow M}$, where $\mathbf{H}(z)$ is given by (7). This concludes the proof. $\bigtriangledown\bigtriangledown\bigtriangledown$

5. CONCLUDING REMARKS

MIMO biorthogonal partners can be found in many signal processing applications including MIMO channel equalization and the multiwavelet theory. The main purpose of this paper is to consider some of the theoretical aspects of MIMO biorthogonal partners. The important issues treated are the existence of biorthogonal partners and their application in the least squares signal approximation.

6. REFERENCES

[1] B. Vrcelj and P. P. Vaidyanathan, "Theory of MIMO biorthogonal partners and their application in channel equalization," to appear in *Proceedings ICC*, Helsinki, Finland, June 2001.

[2] P. P. Vaidyanathan and B. Vrcelj, "Biorthogonal partners and applications," to appear in *IEEE Trans. Signal Processing*.

[3] P. P. Vaidyanathan, *Multirate Systems and Filter Banks,* Prentice Hall, Inc., Englewood Cliffs, N.J.,1993.

[4] S. Mallat, *A Wavelet Tour of Signal Processing,* Academic Press, London, 1998.

[5] M. Unser, A. Aldroubi and M. Eden, "B-spline signal processing: Part I - Theory," *IEEE Trans. Signal Processing*, vol. 41, pp. 821-833, Feb. 1993.

[6] A. Aldroubi and M. Unser, "Oblique projections in discrete signal subspaces of ℓ_2 and the wavelet transform," Proc. SPIE, vol. 2303, *Wavelet appl. in signal and image proc., II*, pp. 36-45, San Diego, CA, 1994.

[7] H. S. Malvar and D. H. Stealin, "Optimal FIR pre- and postfilters for decimation and interpolation of random signals," *IEEE Trans. Comm.*, vol. 36, pp. 67-74, Jan. 1988.

[8] X.-G. Xia, J. S. Geronimo, D. P. Hardin and B. W. Suter, "Design of prefilters for discrete multiwavelet transforms," *IEEE Trans. Signal Processing*, vol. 44, pp. 25-35, Jan. 1996.