BIORTHOGONAL PARTNERS AND APPLICATIONS

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Abstract. Two digital filters $H(z)$ and $F(z)$ are said to be biorthogonal partners of each other if their cascade $H(z)F(z)$ satisfies the Nyquist or zero-crossing property. Biorthogonal partners arise in many different contexts such as filter bank theory, exact and least squares digital interpolation, and multiresolution theory. They also play a central role in the theory of equalization, especially, fractionally spaced equalizers in digital communications. In this paper we first develop several theoretical properties of biorthogonal partners. We also develop conditions for the existence of biorthogonal partners and FIR biorthogonal pairs, and establish the connections to the Riesz basis property. We then explain how these results play a role in many of the above mentioned applications.[1]

1. INTRODUCTION

Two digital filters $H(z)$ and $F(z)$ are said to be biorthogonal partners of each other if their cascade $H(z)F(z)$ satisfies the Nyquist or zero-crossing property. Biorthogonal partners[2] arise in many different contexts such as filter bank theory [1], [23], [27], exact and least-squares digital interpolation [20], sampling theory [22], and multiresolution theory [10]. They also play a central role in the theory of equalization, especially, fractionally spaced equalizers in digital communications [17]. In this paper we first develop several theoretical properties of biorthogonal partners. We then explain the above mentioned applications which use this concept directly or indirectly.

1.1. Outline and Relation to Past Work

The paper contains several new results, and new proofs of well known results. One main contribution here is to glue together certain widely known ideas in a unified manner under one cover. In Sec. 2 we introduce the precise definition of biorthogonal partners. We derive a general closed form expression for a filter $H(z)$ to be a biorthogonal partner of $F(z)$. We also develop a set of necessary and sufficient conditions on an FIR or IIR transfer function $F(z)$ such that there exists an FIR biorthogonal partner $H(z)$. This section also

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[2]The term "biorthogonal partner" has not been used in the past. We use it here because of the frequent need for a descriptive term.

provides a deeper discussion on the existence of biorthogonal partners. In Sec. 3 we study the application of these ideas in digital interpolation. This application also reveals the conditions on $F(z)$ which allow the existence of a biorthogonal partner. It also places in evidence the connection to linear independence and Riesz basis property of the shifted impulse responses $\{f(n-Mk)\}$ of $F(z)$. This work is closely related to the concept of oblique projections studied extensively by Aldroubi et al., and Cohen et al. [2– 5].

Section 5 reviews the role of biorthogonal partners in the least squares approximation of signals using interpolation models. While this idea originated in the context of spline interpolation [7], its efficient implementation became possible because of the work by Unser et al [20, 21] who developed fundamental digital filter structures for efficient implementation of the same. Applications of biorthogonal partners in the interpolation of signals based on continuous time models (e.g., spline models [13], [19]) is also discussed in Sec. 6. We also show that an all-FIR spline interpolation is sometimes possible unlike the more well-known methods of Unser et al. which use a IIR/FIR combination [19]. The role of biorthogonal partners in multiresolution theory is described in Sec. 6.3. We show in particular that an FIR method for the computation of multiresolution coefficients is possible, without resorting to the traditional high degree of oversampling. Finally in Sec. 7 we review applications in the theory of fractionally spaced equalizers for digital communications [12], [17].

1.2. Notations

Unless mentioned otherwise, all notations are as in [23]. We use the notations $[x(n)]_{\downarrow M}$ and $[X(z)]_{\downarrow M}$ to denote the decimated version $x(Mn)$ and its z-transform. The expanded version

$$\begin{cases} x(n/M) & n = \text{mul of } M, \\ 0 & \text{otherwise} \end{cases}$$

is similary denoted by $[x(n)]_{\uparrow M}$, and its z-transform $X(z^M)$ denoted by $[X(z)]_{\uparrow M}$. Notice that

$$[X(z^M)Y(z)]_{\downarrow M} = X(z)\,[Y(z)]_{\downarrow M} \qquad \text{so that} \qquad [X(z^M)Y(z)]_{\downarrow M \uparrow M} = X(z^M)\,[Y(z)]_{\downarrow M \uparrow M}$$

The tilde notation $\widetilde{F}(z)$ is defined by $\widetilde{F}(z) = F^*(1/z^*)$ so that on the unit circle $\widetilde{F}(z) = F^*(e^{j\omega})$. Thus $\widetilde{F}(z)F(z)$ evaluated on the unit circle is the magnitude square function. In situations where the z-transform does not exist in the conventional sense (e.g., ideal filters), the notation z stands for $e^{j\omega}$ so that $H(z)$ is the frequency response $H(e^{j\omega})$.

2. BIORTHOGONAL PARTNERS: DEFINITION AND PROPERTIES

Two transfer functions $F(z)$ and $H(z)$ are said to form a **biorthogonal pair** with respect to an integer M if

$$\Big[H(z)F(z)\Big]_{\downarrow M} = 1 \tag{1}$$

We say that $H(z)$ is the **biorthogonal partner** (or just partner) of $F(z)$. Notice that if M is changed, the two filters may not remain partners. The term "with respect to M" is usually understood from the context and is never mentioned unless there is a possible confusion. Evidently $H(z)$ and $F(z)$ can be interchanged without altering this property. Eqn. (1) is equivalent to the statement that the impulse response $p(n)$ of the product filter $P(z) = H(z)F(z)$ satisfies the **Nyquist(M)** condition

$$p(Mn) = \delta(n)$$

That is, the M-fold decimation of $p(n)$ yields the impulse $\delta(n)$ (Fig. 1). We can regard $H(z)$ and $F(z)$ as any pair that defines a **factorization** of a Nyquist(M) filter $P(z)$.

Notice that every pair of filters $\{H_k(z), F_k(z)\}$ in an M-channel perfect reconstruction (PR or biorthogonal) filter bank satisfies this condition. This is also the reason for the phrase "biorthogonal pair". Recall that the "multirate" system shown in Fig. 2(a) is just an LTI (single rate) system with transfer function $[H(z)F(z)]_{\downarrow M} = 1$ (see "polyphase identity" [23]). Thus the implication of (1) is that the system shown in Fig. 2(a) is an identity system. That is, the decimation filter ($H(z)$ followed by $\downarrow M$) is diagrammatically the right inverse of the interpolation filter ($\uparrow M$ followed by $F(z)$) (Fig. 2(b)).

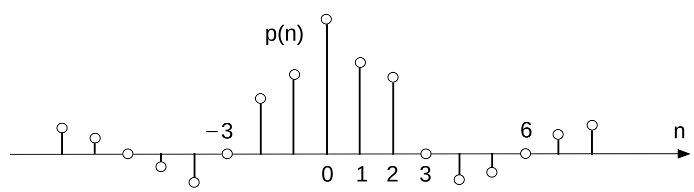

Figure 1. Nyquist(M) property of $P(z)$ demonstrated for $M = 3$.

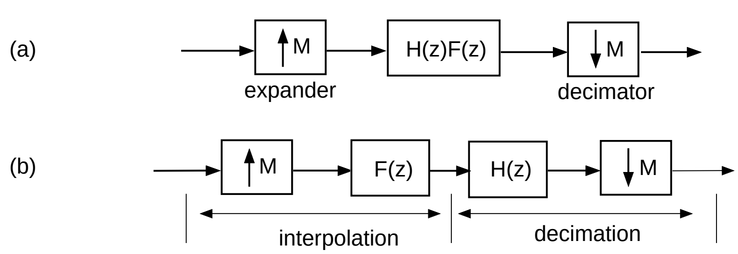

Figure 2. Interpretation of biorthogonal partners $H(z), F(z)$ in terms of signal flowgraphs.

Given a transfer function $F(z)$ and the integer M, does there always exist a biorthogonal partner $H(z)$? When is it unique? If $F(z)$ is FIR, then under what conditions does there exist an FIR biorthogonal partner $H(z)$? For rational $F(z)$ can we get FIR partners? In this section we answer these questions.

2.1. General Expression

We first derive a general expression for $H(z)$ in terms of $F(z)$. In the following theorem note that the notation $[G(z)F(z)]_{\downarrow M\uparrow M}$ stands for $Q(z^M)$ where $Q(z)\stackrel{\Delta}{=}[G(z)F(z)]_{\downarrow M}$.

Theorem 1. General form of biorthogonal partner. The transfer function $H(z)$ satisfies $[H(z)F(z)]_{\downarrow M} = 1$ if and only if it can be expressed in the form

$$H(z) = \frac{G(z)}{\big[G(z)F(z)\big]_{\downarrow M\uparrow M}} \qquad (2)$$

for some transfer function $G(z)$. $\diamondsuit$

Proof. Given a transfer function $H(z)$ of the above form, we have

$$\Big[H(z)F(z)\Big]_{\downarrow M} = \left[\frac{G(z)F(z)}{\big[G(z)F(z)\big]_{\downarrow M\uparrow M}}\right]_{\downarrow M} = \frac{\big[G(z)F(z)\big]_{\downarrow M}}{\big[G(z)F(z)\big]_{\downarrow M}} = 1$$

which proves the "if part". Conversely, suppose $H(z)$ is such that $[H(z)F(z)]_{\downarrow M} = 1$. Consider the interpolation scheme shown in Fig. 3(a) where $x(n)$ is an arbitrary input and $g(n)$ the output of $H(z)$. If we cascade $F(z)$ and the decimator $\downarrow M$ as shown in the figure, then the output is $x(n)$ because $[H(z)F(z)]_{\downarrow M} = 1$. The important point is that, this also means that the signal $g(n)$ input to the system of Fig. 3(b) comes out as $g(n)$. This is because $g(n)$ by definition is the output of the left half in Fig. 3(a) driven by $x(n)$. Thus

$$[G(z)F(z)]_{\downarrow M\uparrow M}H(z) = G(z)$$

which indeed can be rewritten as Eq. (2). $\bigtriangledown\bigtriangledown\bigtriangledown$

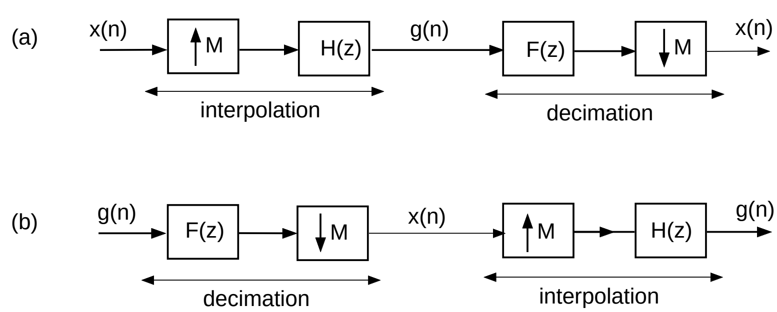

Figure 3. Pertaining to the proof of Theorem 1.

Notice in the proof that since $X(z) = [G(z)F(z)]_{\downarrow M}$ is arbitrary, it can be chosen so that the denominator of (2) is nonzero for all $z = e^{j\omega}$. In general a biorthogonal partner may or may not exist, and when it exists it may not be unique. It follows from Theorem 1 that a stable biorthogonal partner can be found if and only if there exists a $G(z)$ such that $[G(e^{j\omega})F(e^{j\omega})]_{\downarrow M}$ is nonzero for all ω. We shall return to a more insightful discussion of existence issues in Sec. 4. Here are some special situations of interest.

1. If $|F(e^{j\omega})|^2 > 0$ for all ω then $H(z) = 1/F(z)$ is theoretically stable (though not necessarily causal). This is conceptually the simplest biorthogonal partner.

2. If $F(z)$ has **unit circle zeros** then $H(z) = 1/F(z)$ is not a stable filter. But we can often get other solutions. For example suppose the decimated version [23]

$$[F(e^{j\omega})]_{\downarrow M} = \frac{1}{M}\sum_{k=0}^{M-1} F(e^{j(\omega-2\pi k)/M})$$

is nonzero for all ω. Then we can set $G(z) = 1$ in Eq. (2) to obtain the stable biorthogonal partner

$$H(z) = \frac{1}{[F(z)]_{\downarrow M\uparrow M}}$$

3. The preceding examples show that the biorthogonal partner $H(z)$ in general is **not unique**. To get yet another solution, consider the filter

$$H(z) = \frac{\widetilde{F}(z)}{[\widetilde{F}(z)F(z)]_{\downarrow M\uparrow M}}$$

Since $\widetilde{F}(z) = F^*(1/z^*)$, we have $\widetilde{F}(e^{j\omega}) = F^*(e^{j\omega})$ so that $\widetilde{F}(z)F(z) = |F(e^{j\omega})|^2$ on the unit circle. The preceding solution is a special case of (2) with $G(z) = \widetilde{F}(z)$, and works as long as $[\widetilde{F}(e^{j\omega})F(e^{j\omega})]_{\downarrow M}$ is nonzero for all ω. This is a "popular solution" in some sense, and is described in greater detail in Sec. 5, in the context of least squares approximations.

4. If $H_1(z)$ and $H_2(z)$ are biorthogonal partners of $F(z)$ then so is the convex combination $\alpha H_1(z) + (1-\alpha)H_2(z)$.

5. Suppose $F(e^{j\omega})$ is nonzero only in a set of measure $< 2\pi/M$ in $0 \leq \omega < 2\pi$ (e.g., $F(z)$ ideal lowpass with total passband width $< 2\pi/M$). Then $H(e^{j\omega})F(e^{j\omega})$ has the same property, so

$$\Big[H(e^{j\omega})F(e^{j\omega})\Big]_{\downarrow M}$$

cannot fill the region $0 \leq \omega < 2\pi$ completely. There **does not exist** a biorthogonal partner for this $F(e^{j\omega})$.

6. If we replace $G(z)$ with $G(z)S(z^M)$ in Eq. (2), then the $S(z)$ part merely cancels and leaves $H(z)$ unchanged. So $G(z)$ is not unique for a given $H(z), F(z)$ pair. If $G_1(z)$ and $G_2(z)$ are two possible choices then we can verify that $G_1(z) = C(z^M)G_2(z)$ for some $C(z)$ (to see this just divide one representation of $H(z)$ by the other).

2.2. Rational And FIR Cases

In practice the situation of most interest is the case where $F(z)$ is rational, that is, $F(z) = A(z)/B(z)$ where $A(z)$ and $B(z)$ are polynomials in z^{-1}. In this case the trivial choice $H(z) = B(z)/A(z)$ gives a rational biorthogonal partner. This is stable (though possibly noncausal) as long as $A(z)$ has no unit circle zeros. If $F(z)$ is FIR, then the trivial choice $H(z) = 1/F(z)$ yields an allpole IIR filter. A question of significant interest is this: if $F(z)$ is FIR, can we obtain an FIR solution $H(z)$? The answer is provided below. This is key to some of the applications described in the next several sections. In fact this theorem is applicable in nearly the same form in the theory of fractionally spaced equalizers [15, 16, 17].

Theorem 2. Existence of FIR partner. Suppose $F(z)$ is FIR. Express it in the polyphase form $F(z) = \sum_{k=0}^{M-1} z^k R_k(z^M)$. Then there exists an FIR filter $H(z)$ such that $[H(z)F(z)]_{\downarrow M} = 1$ if and only if the gcd (greatest common divisor) $C(z)$ of the M polyphase components $\{R_k(z)\}$ is trivial, i.e., has the form $C(z) = cz^{-N}$ for some constant c and integer N. $\diamondsuit$

Example 1. Thus the gcd should be no more sophisticated than a delay. Given an arbitrary FIR transfer function $F(z)$, this gcd-condition is nearly always satisfied. For example let

$$F(z) = z^{-1} + 4z^{-2} + z^{-3}$$

and $M = 2$. We can write $F(z) = 4z^{-2}+z(z^{-2}+z^{-4})$, which shows that $R_0(z) = 4z^{-1}$ and $R_1(z) = z^{-1}+z^{-2}$. These two polynomials have no common factors other than delays and constant so that the conditions of the theorem are satisfied. Indeed, we can readily verify that the FIR filter $H(z) = z^3 - 0.25z^4$ is an FIR biorthogonal partner of $F(z)$.

Proof of Theorem 2. If the polyphase components $\{R_k(z)\}$ have gcd equal to unity then there exist FIR filters $\{E_k(z)\}$ such that $\sum_{k=0}^{M-1} E_k(z)R_k(z) = 1$. These can be constructed using a generalization of Euclid's algorithm [23]. If the gcd is cz^{-N} then the same is true because the constant c and z^{-N} can be absorbed into $E_k(z)$ anyway. Now define $H(z)$ to be the FIR filter

$$H(z) = \sum_{k=0}^{M-1} z^{-k} E_k(z^M)$$

Then $H(z)F(z)$ has the form $\sum_{i=0}^{M-1} z^i S_i(z^M)$ where $S_0(z) = \sum_{k=0}^{M-1} E_k(z)R_k(z)$. Thus

$$\Big[H(z)F(z)\Big]_{\downarrow M} = S_0(z) = \sum_{k=0}^{M-1} E_k(z)R_k(z) = 1$$

which shows that $H(z)$ is an FIR biorthogonal partner! Conversely suppose there exists an FIR biorthogonal partner $H(z)$. Defining its polyphase components as $E_k(z)$ (i.e., $H(z) = \sum_{k=0}^{M-1} z^{-k} E_k(z^M)$) we have

$[H(z)F(z)]_{\downarrow M} = \sum_{k=0}^{M-1} E_k(z)R_k(z) = 1$. If the gcd of $\{R_k(z)\}$ is $C(z)$ then this can be written as

$$[H(z)F(z)]_{\downarrow M} = C(z) \sum_{k=0}^{M-1} E_k(z)\widehat{R}_k(z) = 1$$

where $E_k(z)$ and $\widehat{R}_k(z)$ are FIR. The preceding equation says that the product of two FIR filters $C(z)$ and $\sum_{k=0}^{M-1} E_k(z)\widehat{R}_k(z)$ is unity. This is not possible unless $C(z)$ has the form cz^{-N}. $\bigtriangledown \bigtriangledown \bigtriangledown$

A block diagram interpretation of this proof is insightful. Recall that the equation $[H(z)F(z)]_{\downarrow M} = 1$ is equivalent to the statement that Fig. 4(a) is an identity system. The polyphase representation of this is shown in Fig. 4(b). We now use the identity shown in Fig. 4(c) (see polyphase identity [23]) to simplify Fig. 4(b) to Fig. 4(d). Thus the equation $[H(z)F(z)]_{\downarrow M} = 1$ is completely equivalent to the statement that the parallel connnection shown in Fig. 4(d) be identity. If $R_k(z)$ are FIR with no overall common factor, then we can indeed find FIR filters $E_k(z)$ so that this is an identity system.

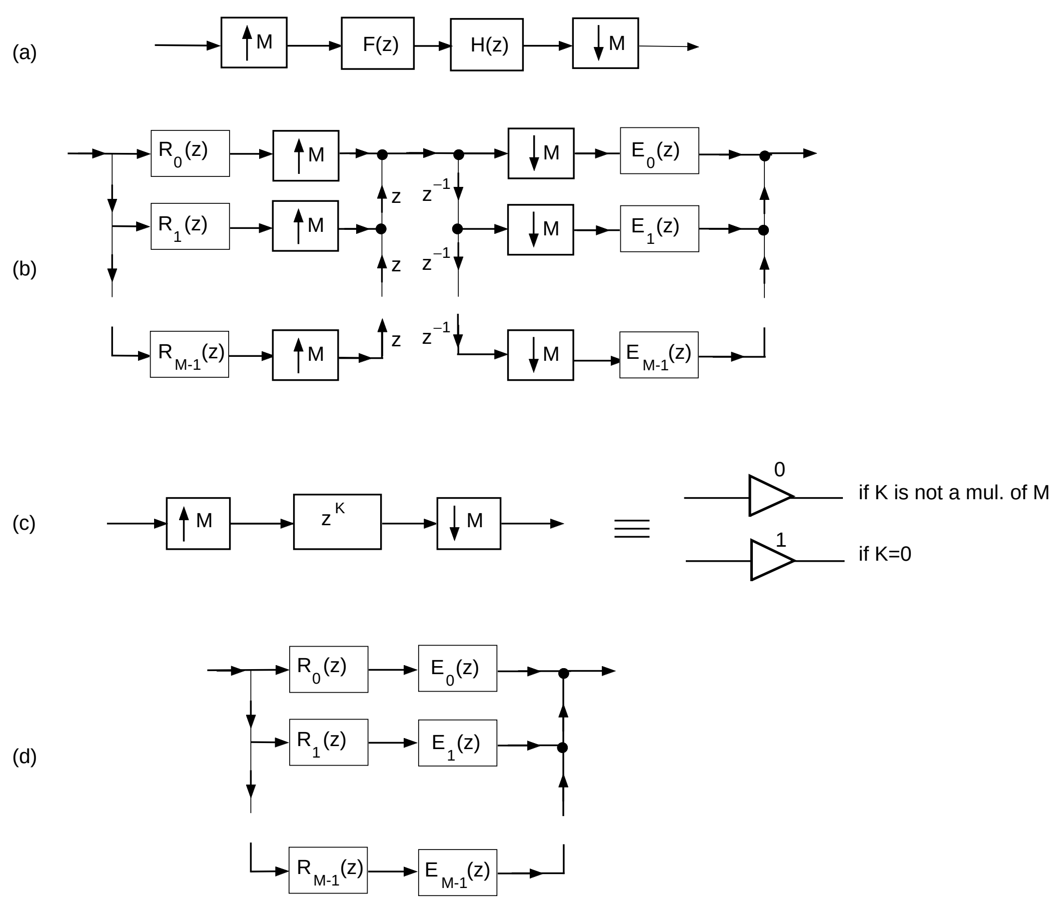

Figure 4. Block diagram interpretation of the proof of Theorem 2. (a) The cascade of interpolator and decimator, (b) redrawing in polyphase form, (c) a basic multirate identity, and (d) simplification of part (b) using the identity.

Corollary 1. FIR partner for IIR filter. Suppose $F(z) = A(z)/B(z)$ which is the most general rational IIR form. Assume the numerator $A(z)$ has an FIR biorthogonal partner $H_a(z)$, so that

$$[H_a(z)A(z)]_{\downarrow M} = 1 \tag{3}$$

Then $H(z) = H_a(z)B(z)$ is an FIR partner for the IIR filter $F(z)$. This is because $H(z)F(z) = H_a(z)A(z)$, so that $[H(z)F(z)]_{\downarrow M} = [H_a(z)A(z)]_{\downarrow M} = 1$ from (3). $\diamondsuit$

3. DISCRETE TIME SIGNAL MODELS

Consider a discrete time signal $x(n)$ that can be modeled as the output of a digital interpolation filter as shown in Fig. 5(a). In this model, $F(z) = \sum_n f(n)z^{-n}$ is a fixed digital filter. We assume $f(n)$ and $c(n)$ are ℓ_2 sequences (finite-energy sequences). By appropriately choosing $c(n)$ we can generate a whole class of signals $x(n)$ like this. This class $\mathcal{F}$ forms a subspace of ℓ_2. Since

$$x(n) = \sum_k c(k)f(n - kM)$$

this is the subspace spanned by the set of sequences $\{\eta_k(n)\} \stackrel{\Delta}{=} \{f(n - kM)\}$. Note that with real-time dimensions the samples of $x(n)$ are spaced apart more closely[3] than those of $c(n)$.

Given a signal $x(n)$ in the subspace $\mathcal{F}$, how do we compute the coefficients $c(n)$ (i.e., the correct driving signal in Fig. 5(a))? Assuming that a biorthogonal partner exists for $F(z)$, all we have to do is to filter $x(n)$ through $H(z)$ and decimate the output as shown in Fig. 5(b). To see this simply note that the output of the decimator is

$$[X(z)H(z)]_{\downarrow M} = [C(z^M)F(z)H(z)]_{\downarrow M} = C(z) \times [F(z)H(z)]_{\downarrow M} = C(z)$$

indeed, because $[F(z)H(z)]_{\downarrow M} = 1$. As seen in Sec. 2.1, the biorthogonal partner $H(z)$ is not unique, but any such $H(z)$ will do. Figure 5(c) shows the decimation system for generating $c(n)$ and the interpolation system for generating $x(n)$ cascaded together. It is clear that $c(n)$ can be regarded as one subband signal of an M-band biorthogonal filter bank with input $x(n)$. If $F(z)$ is FIR with an FIR biorthogonal partner $H(z)$, all computations are FIR based.

[3] In general $c(n)$ is not a subsampled or decimated version of $x(n)$. However the model allows us to recover $x(n)$ from its M-fold decimated version $x(Mn)$ under mild conditions [25, 26].

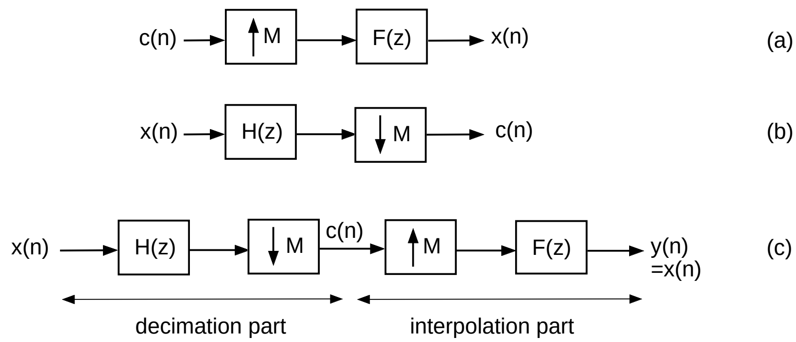

Figure 5. (a) A signal model, (b) computation of the coefficients $c(n)$ in the model for $x(n)$, and (c) the complete system looking like a subband channel in an M-band filter bank.

Lemma 1. Consider the interconnection of Fig. 6 and assume $F(z)$ is rational (FIR or IIR). If $H(z)$ is such that for any input $x(n) \in \mathcal{F}$ (i.e., any $x(n)$ with $X(z) = C(z^M)F(z)$) the final output $y(n)$ is exactly equal to $x(n)$, then $H(z)$ is necessarily a biorthogonal partner of $F(z)$. The assertion is **not true** when nonrational $F(z)$ (e.g., ideal brickwall filters) is allowed. $\diamondsuit$

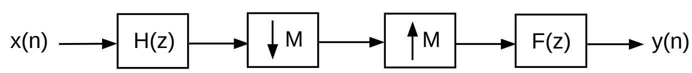

Figure 6. Pertaining to Lemma 1.

Proof. The filter $H(z)$ in the Lemma is such that if an input of the form $X(z) = C(z^M)F(z)$ is applied at the left in Fig. 6 then the output of $F(z)$ is also $X(z)$. That is, the following equation holds:

$$\Big([C(z^M)F(z)H(z)]_{\downarrow M\uparrow M}\Big) \times F(z) = C(z^M)F(z)$$

Using standard multirate identities, we can factor out $C(z^M)$ from the left side and write this as

$$C(z^M)\Big([F(z)H(z)]_{\downarrow M\uparrow M}\Big) \times F(z) = C(z^M)F(z)$$

If this holds for all $C(z)$ such that $c(n) \in \ell_2$, we can cancel $C(z^M)$ from both sides. Since $F(z)$ is rational, it can be cancelled as well, proving that $[F(z)H(z)]_{\downarrow M} = 1$, that is, $H(z)$ is a biorthogonal partner of $F(z)$.

If $F(z)$ is allowed to be nonrational, the cancellation step is not valid. In fact we can produce a counter example showing that the same assertion is not true: let $F(e^{j\omega})$ be lowpass as in Fig. 7(a). For any $H(e^{j\omega})$, the support of $H(e^{j\omega})F(e^{j\omega})$ is restricted to $|\omega| < \pi/4$ so that $[H(e^{j\omega})F(e^{j\omega})]_{\downarrow 2}$ is zero for $\pi/2 < \omega < 3\pi/2$ as demonstrated in Fig. 7(b). The filter $F(z)$ therefore has no biorthogonal partner for $M = 2$. However the choice $H(e^{j\omega}) = 2F(e^{j\omega})$ is such that if $x(n) \in \mathcal{F}$ is applied to $H(z)$ in Fig. 6 the output of $F(z)$ is indeed $x(n)$. This is because, even though $[F(z)H(z)]_{\downarrow 2} \neq 1$, we have $F(z)[F(z)H(z)]_{\downarrow 2\uparrow 2} = F(z)$ in this example, so that

$$Y(z) = \Big([C(z^2)F(z)H(z)]_{\downarrow 2\uparrow 2}\Big) \times F(z) = C(z^2)\Big([F(z)H(z)]_{\downarrow 2\uparrow 2}\Big) \times F(z) = C(z^2)F(z) = X(z)$$

indeed. $\bigtriangledown \bigtriangledown \bigtriangledown$

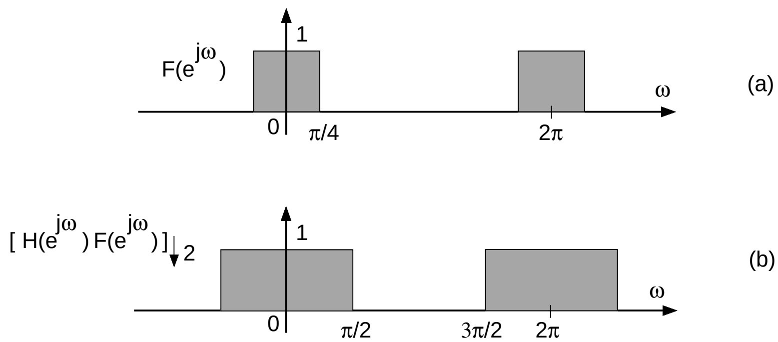

Figure 7. (a) The filter $F(e^{j\omega})$ in the model, and (b) $[H(e^{j\omega})F(e^{j\omega})]_{\downarrow 2}$ for $H(e^{j\omega}) = 2F(e^{j\omega})$.

4. EXISTENCE OF BIORTHOGONAL PARTNERS

From Theorem 1 we know that $F(z)$ has a stable biorthogonal partner if there exists $G(z)$ such that $[G(z)F(z)]_{\downarrow M}$ has no unit circle zeros. It is insightful to look at the existence issue in different ways as we shall do in this section.

Lemma 2. Suppose $F(z)$ has a biorthogonal partner $H(z)$ so that $[H(z)F(z)]_{\downarrow M} = 1$. Then the signals

$$\eta_k(n) \stackrel{\Delta}{=} f(n - kM) \tag{4}$$

are linearly independent. $\diamondsuit$

Proof. If $\eta_k(n)$ are linearly dependent, then there exists a sequence $c(n)$, not zero for all n, such that $\sum_k c(k)f(n - kM) = 0$. In z-transform notation this means $C(z^M)F(z) = 0$. Thus

$$0 = \Big(C(z^M)F(z)H(z)\Big)_{\downarrow M} = C(z) \times \Big(F(z)H(z)\Big)_{\downarrow M} = C(z)$$

because $[F(z)H(z)]_{\downarrow M} = 1$. This shows that $C(z) = 0$, contradicting the assumption that $c(n)$ is not the zero sequence. $\bigtriangledown \bigtriangledown \bigtriangledown$

Corollary 2. In Fig. 6 suppose $H(z)$ is such that any input $x(n) \in \mathcal{F}$ produces the output $y(n) = x(n)$. In general this does not mean that $H(z)$ is a biorthogonal partner because such a partner may not even exist (Lemma 1). However, if $F(z)$ does have a biorthogonal partner then $H(z)$ is such a partner. $\diamondsuit$

Proof. Choose $X(z) = F(z) \in \mathcal{F}$. Then $Y(z) = F(z)$ as well so that

$$F(z) = \Big([F(z)H(z)]_{\downarrow M \uparrow M}\Big)F(z)$$

Thus $(D(z^M) - 1)F(z) = 0$ where $D(z) \stackrel{\Delta}{=} [F(z)H(z)]_{\downarrow M}$. If there exists a biorthogonal partner for $F(z)$ then by the linear independence asseted by Lemma 2 it follows that $D(z^M) - 1 = 0$, that is, $D(z) = [F(z)H(z)]_{\downarrow M} = 1$. In other words $H(z)$ is a biorthogonal partner of $F(z)$. $\bigtriangledown \bigtriangledown \bigtriangledown$

Lemma 3. Suppose $F(z)$ is stable (i.e., $\sum_n |f(n)| < \infty$) and is such that the set of sequences $\{\eta_k(n)\}$ defined in (4) is linearly independent in the sense that $\sum_k c(k)f(n - kM)$ cannot be arbitrarily small for any $c(n)$ with fixed nonzero energy. More precisely, suppose there exists $A > 0$ such that[4]

$$\sum_n \Big|\sum_k c(k)f(n - kM)\Big|^2 \geq A \sum_n |c(n)|^2 \tag{5}$$

Then $F(z)$ has a biorthogonal partner $H(z)$. $\diamondsuit$

Proof. We will show that Eq. (5) implies that $(|F(e^{j\omega})|^2)_{\downarrow M} > 0$ for all ω. Then the filter

$$H(e^{j\omega}) \stackrel{\Delta}{=} \frac{F^*(e^{j\omega})}{(|F(e^{j\omega})|^2)_{\downarrow M \uparrow M}}$$

is a biorthogonal partner (set $G(e^{j\omega}) = F^*(e^{j\omega})$ in Theorem 1). To show that $(|F(e^{j\omega})|^2)_{\downarrow M} > 0$ assume the contrary, that is, $(|F(e^{j\omega_1})|^2)_{\downarrow M} = 0$ for some ω_1. That is,

$$\sum_{k=0}^{M-1} |F(e^{j(\frac{\omega_1 + 2\pi k}{M})})|^2 = 0$$

Since each term in the above summation is nonnegative, this implies $F(e^{j(\frac{\omega_1+2\pi k}{M})}) = 0$ for each k. That is,

$$F(e^{j(\omega_0 + \frac{2\pi k}{M})}) = 0, \quad 0 \leq k \leq M - 1,$$

where $\omega_0 = \omega_1/M$. Now consider the interpolation scheme in Fig. 8(a). Choose $c(n)$ such that $C(e^{j\omega})$ is the pulse shown in Fig. 8(b), with energy concentrated around ω_1. Note that the energy $\sum_n |c(n)|^2 = \int_0^{2\pi} |C(e^{j\omega})|^2 d\omega/2\pi = 1$ for any ϵ. The output of the expander, which is $C(e^{j\omega M})$ evidently has the same energy, but it is distributed around the frequencies $(\omega_0 + 2\pi k/M)$. If we make ϵ arbitrarily small, the energy of $x(n)$, namely,

$$\sum_n \Big|\sum_k c(k)f(n - kM)\Big|^2$$

is concentrated more and more around the zeros of $F(e^{j\omega})$. Since $\sum_n |f(n)| < \infty$, the response $F(e^{j\omega})$ is a continuous function and so is $|F(e^{j\omega})|$. So the energy of $x(n)$ can be made arbitrarily small, though the

[4]Note that this is one of the two conditions for $\{f(n - kM)\}$ to be a Riesz basis [6,24].

energy of $c(n)$ remains unity. This means there cannot exist $A > 0$ satisfying (5). Summarizing, if Eq. (5) is satisfied, then $(|F(e^{j\omega})|^2)_{\downarrow M} > 0$ for all ω. $\bigtriangledown\bigtriangledown\bigtriangledown$

Connection to Riesz basis. The condition that $F(z)$ be stable in Lemma 3 implies in particular that $|F(e^{j\omega})|^2 < B$ for some $B < \infty$. So $\int |X(e^{j\omega})|^2 d\omega/2\pi = \int |F(e^{j\omega})C(e^{j\omega M})|^2 d\omega/2\pi \leq B \int |C(e^{j\omega})|^2 d\omega/2\pi$, Since $x(n) = \sum_k c(k)f(n-kM)$ this implies by Parseval's relation that

$$\sum_n \Big|\sum_k c(k)f(n-kM)\Big|^2 \leq B\sum_n |c(n)|^2$$

Thus, under the conditions of Lemma 3 both the preceding inequality and (5) are true, that is

$$A\sum_n |c(n)|^2 \leq \sum_n \Big|\sum_k c(k)f(n-kM)\Big|^2 \leq B\sum_n |c(n)|^2$$

where $A > 0$ and $B < \infty$. This is precisely the definition of a Riesz basis. That is, under the conditions of Lemma 3, $\{f(n-kM)\}$ is a Riesz basis for the subspace $\mathcal{F}$ defined at the beginning of Sec. 3.

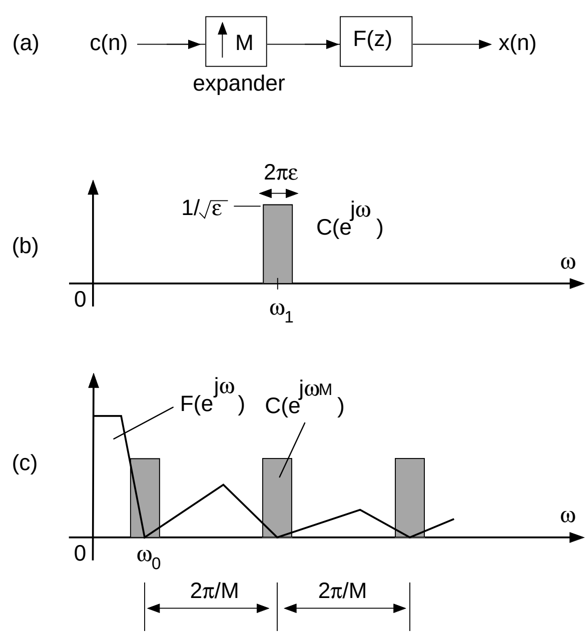

Figure 8. Pertaining to the proof of Lemma 3. (a) An interpolation filter, (b) a narrowband unit-energy input, and (c) filter $F(e^{j\omega})$ and its input $C(e^{j\omega M})$.

The main points of the preceding two lemmas can be usefully summarized as follows: a digital filter $F(z)$ has a biorthogonal partner (with respect to integer M) if and only if $\eta_k(n) \stackrel{\Delta}{=} f(n-kM)$ are linearly independent. From the technique of the proof of Lemma 3 we readily obtain the following:

Theorem 3. Existence of biorthogonal partners. The filter $F(z)$ has a biorthogonal partner if and only if $(|F(e^{j\omega})|^2)_{\downarrow M} > 0$ for all ω. Thus, if there is a biorthogonal partner, then in particular the choice $H(e^{j\omega}) = F^*(e^{j\omega})/[(|F(e^{j\omega})|^2)_{\downarrow M\uparrow M}]$ will work. Moreover define the new filters

$$H_1(e^{j\omega}) = \frac{F^*(e^{j\omega})}{\Big(|(F(e^{j\omega})|^2)_{\downarrow M\uparrow M}\Big)^{1/2}}, \qquad F_1(e^{j\omega}) = \frac{F(e^{j\omega})}{\Big((|F(e^{j\omega})|^2)_{\downarrow M\uparrow M}\Big)^{1/2}}$$

Then $H_1(e^{j\omega})$ is a biorthogonal partner of $F_1(e^{j\omega})$ and moreover the set of signals $\{\eta_k(n)\}\stackrel{\Delta}{=}\{f_1(n-kM)\}$ forms an **orthonormal basis** for the space $\mathcal{F}$ spanned by $\{f(n-kM)\}$. $\diamondsuit$

Proof. If $(|F(e^{j\omega})|^2)_{\downarrow M} > 0$ then we can define $H(e^{j\omega}) = F^*(e^{j\omega})/(|F(e^{j\omega})|^2)_{\downarrow M\uparrow M}$ and this is a biorthogonal partner. Conversely if $F(z)$ has a biorthogonal partner then the signals $\eta_k(n) = f(n-kM)$ are linearly independent (Lemma 2). This means $(|F(e^{j\omega})|^2)_{\downarrow M}$ cannot be zero for any ω (otherwise we can create an annihilating input as in the proof of Lemma 3 violating linear independence). The biorthogonality of $H_1(z)$ and $F_1(z)$ follows readily. Moreover $|F_1(e^{j\omega})|^2_{\downarrow M} = 1$, that is, $|F_1(e^{j\omega})|^2$ is Nyquist(M). In the time domain this means $\sum_n f_1(n)f_1^*(n-kM) = \delta(k)$ which is equivalent to the orthonormality of $\{f_1(n-kM)\}$. $\bigtriangledown\bigtriangledown\bigtriangledown$

For rational filters, we can replace $(|F(e^{j\omega})|^2)_{\downarrow M})^{1/2}$ in the denominator of $F_1(e^{j\omega})$ with a spectral factor of $(\widetilde{F}(z)F(z))_{\downarrow M}$ to obtain an orthonormal basis. We then set $H_1(z) = \widetilde{F}_1(z)$ as in any orthonormal filter bank. Notice that the orthogonalized filter $F_1(z)$ can be written as $F_1(z) = F(z)/D(z^M)$ where $D(z)$ is a spectral factor of $(\widetilde{F}(z)F(z))_{\downarrow M}$. Thus

$$X(z) = C(z^M)F(z) = C(z^M)D(z^M)F_1(z) = C_1(z^M)F_1(z).$$

That is, $x(n) = \sum_k c_1(k)f_1(n-kM)$, so the coefficients of expansion in the new basis are $c_1(n)$.

Corollary 3. FIR case. For FIR $F(z)$ there exists a biorthogonal partner if and only if $F(z)$ is free from factors of the form $(1 - z^{-M}e^{j\omega_1})$. These factors represent a set of M zeros spaced uniformly on the unit circle. $\diamondsuit$

Proof. If $F(z)$ has the factor $(1 - z^{-M}e^{j\omega_1})$ then $(|F(e^{j\omega})|^2)_{\downarrow M} = |1 - e^{-j(\omega-\omega_1)}|^2 \times |F_1(e^{j\omega})|^2$ for some FIR $F_1(z)$. This vanishes at $\omega = \omega_1$, violating $(|F(e^{j\omega})|^2)_{\downarrow M} > 0$. Conversely if $(|F(e^{j\omega})|^2)_{\downarrow M} > 0$ is violated, there exists ω_0 such that $F(e^{j(\omega_0+2\pi k/M)}) = 0$ for all k (see proof of Lemma 3). That is, $F(z)$ has the factor

$$\prod_{k=0}^{M-1}(1 - z^{-1}e^{j\omega_0}e^{j2\pi k/M})$$

which can be rewritten as $(1 - z^{-M}e^{j\omega_1})$ for $\omega_1 = M\omega_0$. Thus the condition $(|F(e^{j\omega})|^2)_{\downarrow M} > 0$ in Theorem 3 is equivalent to the nonexistence of factors of the form $(1 - z^{-M}e^{j\omega_1})$. $\bigtriangledown\bigtriangledown\bigtriangledown$

Oblique projections. Some of the results in this section can also be found in the mathematics literature, in the more general setting of oblique projections [2–5]. The article by Aldroubi and Unser [3] is especially insightful for the discrete time case. The reader should read Theorems 1 and 2 of [3] to see the connection to this section. The papers by Cohen et al. [4, 5] address many deep issues in the continuous time case.

5. APPLICATION IN LEAST SQUARES INTERPOLATION

Consider the class $\mathcal{F}$ of discrete time signals $y(n)$ which can be modeled as the output of a fixed interpolation filter $F(z)$ as shown in Fig. 9. We shall refer to this as the **interpolation-filter model**. One situation where this model arises is in sampling theory: we can reconstruct $y(n)$ from the decimated version $y(nM)$ under some mild conditions [25, 26]. Given an arbitrary signal $x(n) \in \ell_2$ suppose we wish to approximate it by the model signal $y(n) \in \mathcal{F}$. This can be done by proper choice of the lower rate signal $c(n)$. Let us say we want $c(n)$ to be chosen such that

$$\sum_n |y(n) - x(n)|^2$$

is minimized. This least squares solution $y(n)$ is nothing but the **orthogonal projection** of $x(n)$ onto $\mathcal{F}$. This problem is related to a number of things in filter bank theory as we shall see. In particular it arises in the context of least square **spline** approximation of **continuous time** L_2 signals as shown by Unser et al. [20]. It also arises in the optimal subband coder problem as shown by Strintzis, see for example [14].

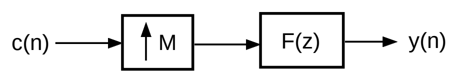

Figure 9. An interpolation-filter model defining a class of signals $y(n)$.

Theorem 4. Given the filter $F(z)$, assume $(|F(e^{j\omega})|^2)_{\downarrow M} > 0$ and define a new filter

$$H(z) = \frac{\widetilde{F}(z)}{\left[\widetilde{F}(z)F(z)\right]_{\downarrow M \uparrow M}} \tag{6}$$

If we pass the given signal $x(n)$ through $H(z)$ and decimate it by M we get the optimal $c(n)$ (Fig. 10(a)). This $c(n)$ can be used to find the least squares approximation $y(n)$. The complete system is shown in Fig. 10(b). The filter (6) is called the (orthogonal) **projection prefilter** corresponding to the interpolation filter $F(z)$. $\diamondsuit$

Proof. The assumption $(|F(e^{j\omega})|^2)_{\downarrow M} > 0$ ensures that the denominator of $H(z)$ does not have unit circle zeros. The squared error can be rewritten in the frequency domain as follows:

$$\sum_n |y(n) - x(n)|^2 = \int_0^{2\pi} |Y(e^{j\omega}) - X(e^{j\omega})|^2 d\omega/2\pi = \int_0^{2\pi} |C(e^{j\omega M})F(e^{j\omega}) - X(e^{j\omega})|^2 d\omega/2\pi$$

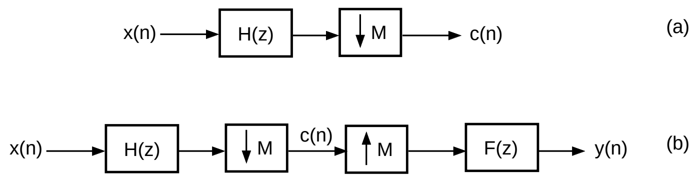

Figure 10. Generating the best approximation $y(n)$ of the given signal $x(n)$. (a) Computing the driving signal $c(n)$, and (b) the complete system.

The aim is to choose $C(e^{j\omega})$ optimally to minimize this. Notice that $C(e^{j\omega M})$ appearing in the integrand has period $2\pi/M$, and can be chosen independently only in the range $0 \le \omega < 2\pi/M$. So let us rewrite

$$\sum_n |y(n) - x(n)|^2 = \int_0^{2\pi/M} \underbrace{\sum_{k=0}^{M-1} \left| C(e^{j\omega M}) F(e^{j(\omega + \frac{2\pi k}{M})}) - X(e^{j(\omega + \frac{2\pi k}{M})}) \right|^2}_{Q(\omega)} d\omega/2\pi$$

For each ω in $0 \le \omega < 2\pi/M$ we can choose $C(e^{j\omega M})$ independently such that the nonnegative integrand $Q(\omega)$ is minimized. Note that $C(e^{j\omega M})$ is independent of the summation index k. Define the vectors

$$\mathbf{a}(\omega) = \begin{bmatrix} X(e^{j\omega}) \\ X(e^{j(\omega + \frac{2\pi}{M})}) \\ \vdots \\ X(e^{j(\omega + \frac{2\pi(M-1)}{M})}) \end{bmatrix}, \quad \mathbf{b}(\omega) = \begin{bmatrix} F(e^{j\omega}) \\ F(e^{j(\omega + \frac{2\pi}{M})}) \\ \vdots \\ F(e^{j(\omega + \frac{2\pi(M-1)}{M})}) \end{bmatrix}$$

The problem is one of minimizing

$$\|C(e^{j\omega M})\mathbf{b}(\omega) - \mathbf{a}(\omega)\|^2 = C^*(e^{j\omega M})\mathbf{b}^\dagger(\omega)\mathbf{b}(\omega)C(e^{j\omega M}) - C^*(e^{j\omega M})\mathbf{b}^\dagger(\omega)\mathbf{a}(\omega) - \mathbf{a}^\dagger(\omega)\mathbf{b}(\omega)C(e^{j\omega M}) + \mathbf{a}^\dagger(\omega)\mathbf{a}(\omega)$$

By using the familiar trick of "completion of squares", this can be rewritten as

$$\|C(e^{j\omega M})\mathbf{b}(\omega) - \mathbf{a}(\omega)\|^2 = \Big(C^*(e^{j\omega M}) - \frac{\mathbf{a}^\dagger\mathbf{b}}{\mathbf{b}^\dagger\mathbf{b}}\Big)\mathbf{b}^\dagger\mathbf{b}\Big(C(e^{j\omega M}) - \frac{\mathbf{b}^\dagger\mathbf{a}}{\mathbf{b}^\dagger\mathbf{b}}\Big) + \mathbf{a}^\dagger\mathbf{a} - \frac{|\mathbf{a}^\dagger\mathbf{b}|^2}{\mathbf{b}^\dagger\mathbf{b}}$$

This shows that the best unique choice of $C(e^{j\omega M})$ is $C(e^{j\omega M}) = \mathbf{b}^\dagger(\omega)\mathbf{a}(\omega)/\mathbf{b}^\dagger(\omega)\mathbf{b}(\omega)$. To rewrite this in terms of filters and multirate building blocks recall that [23] $[S(e^{j\omega})]_{\downarrow M} = \sum_{k=0}^{M-1} S(e^{j(\omega+2\pi k)/M})/M$. Thus

$$\mathbf{b}^\dagger(\omega)\mathbf{a}(\omega) = \sum_{k=0}^{M-1} F^*(e^{j(\omega + \frac{2\pi k}{M})})X(e^{j(\omega + \frac{2\pi k}{M})}) = M\Big[F^*(e^{j\omega})X(e^{j\omega})\Big]_{\downarrow M \uparrow M}$$

$$\mathbf{b}^\dagger(\omega)\mathbf{b}(\omega) = \sum_{k=0}^{M-1} |F(e^{j(\omega + \frac{2\pi k}{M})})|^2 = M\Big[|F(e^{j\omega})|^2\Big]_{\downarrow M \uparrow M}$$

The optimal $C(e^{j\omega M})$ is therefore $C(e^{j\omega M}) = \Big([F^*(e^{j\omega})X(e^{j\omega})]_{\downarrow M \uparrow M}\Big) \Big/ \Big([|F(e^{j\omega})|^2]_{\downarrow M \uparrow M}\Big)$. That is,

$$C(z) = \frac{[\widetilde{F}(z)X(z)]_{\downarrow M}}{[\widetilde{F}(z)F(z)]_{\downarrow M}} = \Bigg[\underbrace{\Bigg(\frac{\widetilde{F}(z)}{[\widetilde{F}(z)F(z)]_{\downarrow M \uparrow M}}\Bigg)}_{H(z)} X(z)\Bigg]_{\downarrow M}$$

That is, $c(n)$ can be generated by filtering $x(n)$ through $H(z)$, and decimating by M (Fig. 10(b)). $\bigtriangledown\bigtriangledown\bigtriangledown$

Theorem 5. Uniqueness of prefilter. For fixed $F(z)$ and $x(n) \in \ell_2$, the least squares approximation $y(n)$ is unique, and so is the generating signal $c(n)$. Next, suppose the prefilter $H(z)$ is such that the output of $F(z)$ is the least squares approximation of $x(n)$ for any choice of the ℓ_2 input $x(n)$ in Fig. 10(b). Then $H(z)$ is unique and is therefore given by the projection prefilter (6). $\diamondsuit$

Proof. The uniqueness of $y(n)$ and $c(n)$ follows from the proof of Theorem 4. Next, let $H(z)$ and $H_1(z)$ be two prefilters and let them both be optimal for all $x(n) \in \ell_2$. Since the optimal $c(n)$ is unique as seen from the proof of Theorem 4, we see that

$$\Big(X(z)[H(z) - H_1(z)]\Big)_{\downarrow M} = 0$$

for all $X(z)$. Choosing $X(z) = z^k$ this implies that the kth polyphase component of $H(z)-H_1(z)$ is zero [23]. Since this is true for all k, we conclude $H(z) - H_1(z) = 0$. So the prefilter is unique. While this argument is elegant, the result of Theorem 5 also follows from the uniqueness of the orthogonal projection operator onto a closed subspace [3]. $\bigtriangledown\bigtriangledown\bigtriangledown$

Remarks

1. ***Partner property.*** Note that $H(z)$ given by (6) is a biorthogonal partner of $F(z)$, that is $[H(z)F(z)]_{\downarrow M} = 1$. This follows from Theorem 1 by setting $G(z) = \widetilde{F}(z)$. The assumption $(|F(e^{j\omega})|^2)_{\downarrow M} > 0$ in Theorem 4 is equivalent to the statement that a biorthogonal partner exists (Theorem 3). Even though the optimal prefilter $H(z)$ generating $y(n)$ is a biorthogonal partner of $F(z)$, we see from Theorem 5 that an **arbitrary biorthogonal partner** of $F(z)$ will not work in the least squares problem. This is unlike in Sec. 3 where the signal $c(n)$ could be produced from $x(n)$ using any biorthogonal partner $H(z)$.

2. ***Orthonormal case.*** If $(\widetilde{F}(z)F(z))_{\downarrow M} = 1$, that is, $\widetilde{F}(z)F(z)$ is Nyquist(M) then the solution for $H(z)$ becomes $H(z) = \widetilde{F}(z)$, that is, $h(n) = f^*(-n)$, which is time reversed conjugation as in matched filtering. Recall that this condition arises in each subband of an orthonormal filter bank. Indeed, the interpolated subband outputs in any orthonormal filter bank represent projections of the input $x(n)$ onto subspaces spanned by the synthesis filter functions $\eta_{ki}(n) = f_i(n - Mk)$.

Example 2.

We now present an example demonstrating various aspects of the least squares interpolation model. Let $M = 2$ and assume $F(z)$ is the first order FIR filter $F(z) = 1 + bz^{-1}$ for some real b. Then $(\widetilde{F}(z)F(z))_{\downarrow 2} =$

$1+b^2$, and Theorem 4 yields

$$H(z) = \frac{\widetilde{F}(z)}{\left[\widetilde{F}(z)F(z)\right]_{\downarrow 2\uparrow 2}} = \frac{1+bz}{1+b^2}$$

Suppose we wish to approximate a finite duration signal $x(n)$ with z-transform $X(z) = 1 + az^{-1}$. Then the optimal $C(z)$ is given by

$$C(z) = \Big(X(z)H(z)\Big)_{\downarrow 2} = \Big(\frac{1+ab+az^{-1}+bz}{1+b^2}\Big)_{\downarrow 2} = \frac{1+ab}{1+b^2}$$

To demonstrate that arbitrary biorthogonal partners may not be optimal, consider the biorthogonal partner of $F(z)$ given by $H_2(z) = 1/F(z) = 1/(1+bz^{-1})$. The decimated subband signal is $[X(z)H_2(z)]_{\downarrow 2} = (1-abz^{-1})/(1-b^2z^{-1})$ and is not the optimal $C(z)$. Consider next a transfer function of the form $E(z) = z^{-2K}(1 - z/a)$ where K is an arbitrary integer. We have

$$(X(z)E(z))_{\downarrow 2} = 0$$

which shows that if $H(z)$ is replaced with $H_1(z) = H(z) + E(z)$ then the output of $F(z)$ in Fig. 10(b) is still the least squares approximation $y(n)$. This shows that the optimum prefilter is **not unique**; in fact $H_1(z)$ is not a even biorthogonal partner! These instances occur if the goal is to make the prefilter work for only some specific choices of $x(n)$. If the prefilter has to work for all $x(n) \in \ell_2$ then (6) is the only choice.

Summary of this section. Here is the summary of what we have shown under the mild assumption that $(|F(e^{j\omega})|^2)_{\downarrow M} > 0$. The least squares approximation $y(n)$ is unique and so is the driving signal $c(n)$. If the prefilter $H(z)$ has to generate the optimal $c(n)$ for all $x(n) \in \ell_2$ then $H(z)$ is the unique filter called the projection prefilter and is given by (6). This prefilter also happens to be a biorthogonal partner of $F(z)$.

6. CONTINUOUS TIME INTERPOLATION MODELS

We now show how the results of earlier sections find application in interpolation based on continous-time models. As a first step we review a well-known linear interpolation model, and its efficient implementation developed in the fundamental work by Unser et al. [19]. An excellent review of sampling in this context was recently given by Unser [22].

6.1. Review

Given a discrete time signal $x_d(n)$ and an arbitrary function $\phi(t)$ we can almost always assume that $x_d(n)$ can be written in the form

$$x_d(n) = \sum_{k=-\infty}^{\infty} c(k)\phi(n-k) \tag{7}$$

for appropriate choice of $c(k)$. This is because the equation is equivalent to $X_d(e^{j\omega}) = C(e^{j\omega})\Phi_{d,1}(e^{j\omega})$ in the frequency domain, where $\Phi_{d,1}(e^{j\omega})$ is the discrete time FT of the sampled sequence $\phi(n)$ (the need for a second subscript "1" will be clear soon). Thus we can calculate $c(k)$ from $x_d(n)$ by inverse digital filtering, that is, $C(e^{j\omega}) = X_d(e^{j\omega})/\Phi_{d,1}(e^{j\omega})$. The only theoretical condition is that $\Phi_{d,1}(e^{j\omega})$ be nonzero for all ω so that $1/\Phi_{d,1}(e^{j\omega})$ represents a stable filter. As we shall see below the stability condition can readily be satisfied in practice.

The preceding observation shows that we can regard $x_d(n)$ as samples of a continuous time signal $x(t)$ which admits the specific model

$$x(t) = \sum_{k=-\infty}^{\infty} c(k)\phi(t-k), \tag{8}$$

the sample spacing being $T = 1$. While true for almost any $\phi(t)$, this is especially useful for certain choices of $\phi(t)$. For example if $\phi(t)$ has smoothness properties such as a certain degree of differentiability everywhere, then we can use this to generate a good interpolated version of $x(n)$. A 256×256 image can be displayed as a 512×512 image in this way (interpolation by two). Smoothness of $\phi(t)$ usually ensures that the interpolated result is visually pleasing (see example below). To see how the model can be used for interpolation notice that the samples of $x(t)$ at a finer spacing $1/L$ are given by

$$x(n/L) = \sum_{k=-\infty}^{\infty} c(k)\phi(n/L-k) = \sum_{k=-\infty}^{\infty} c(k)\phi\Big(\frac{n-kL}{L}\Big) = \sum_{k=-\infty}^{\infty} c(k)\phi_{d,L}(n-kL) \tag{9}$$

where $\phi_{d,L}(n) = \phi(n/L)$ is the filter obtained by sampling $\phi(t)$ at a **finer** spacing of $1/L$. In summary, we can reconstruct the finer samples $x(n/L)$ from $x(n)$ as shown in Fig. 11: first pass $x(n)$ through the digital prefilter

$$H(e^{j\omega}) = 1/\Phi_{d,1}(e^{j\omega}). \tag{10}$$

This gives $c(n)$. Then use the L-fold upsampler or expander [23], followed by the interpolation filter $\Phi_{d,L}(z) = \sum_n \phi_{d,L}(n)z^{-n}$. We see that the interpolation from $x(n)$ to $x(n/L)$ can be done entirely digitially. The function $\phi(t)$ is often chosen as a **spline** function, the use of cubic splines being especially common. For the rest of the section we shall frequently use the following notations:

$$\phi_{d,L}(n) \stackrel{\Delta}{=} \phi(n/L), \quad \Phi_{d,L}(z) = \sum_n \phi_{d,L}(n)z^{-n}$$

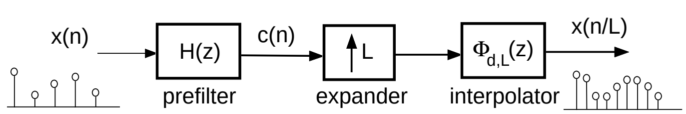

Figure 11. Interpolation of a signal $x(n)$ with digital filters. The signal is assumed to have a continuous time model $x(t) = \sum_k c(k)\phi(t-k)$.

In many practical systems the function $\phi(t)$ is of finite duration. This makes $\phi_{d,1}(n)$ FIR which means that $H(e^{j\omega}) = 1/\Phi_{d,1}(e^{j\omega})$ is IIR. In general this IIR filter may not have all poles inside the unit circle (this problem arises when $\phi(t)$ is a spline [19]). An Nth order B-spline is nothing but the convolution of the pulse function

$$p(t) = \begin{cases} 1 & 0 \leq t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

with itself N times so that the Fourier transform of the Nth order B-spline is[5]

$$\Phi(j\omega) = e^{-j\omega(N+1)/2} \left(\frac{\sin(\omega/2)}{(\omega/2)} \right)^{N+1}$$

It can be verified [13] that the corresponding time domain expression is

$$\phi(t) = \sum_{k=0}^{N+1} \binom{N+1}{k} (-1)^k \frac{(t-k)^N}{N!} \mathcal{U}(t-k)$$

where $\mathcal{U}(t)$ is the unit step function. The beauty of an Nth order spline is that it is **continuously differentiable** $N-1$ times everywhere (i.e., the $(N-1)$th derivative exists and is continuous). Moreover the Nth derivative is a piecewise constant. The differentiability is true even at the end points for finite duration splines such as the B-spline which has duration $N+1$. In fact Nth order splines are **polynomials** of degree N between integers. These polynomial pieces are glued together such that they are sufficiently differentiable even at the integers.

For example assume that $\phi(t)$ is the 3rd order spline (or cubic spline) which is popularly used in image interpolation [19]. In this case it can be shown that the sampled version $\phi(n)$ has z-transform $\Phi_{d,1}(z) = (z^{-1} + 4z^{-2} + z^{-3})/6$ which is FIR. So the prefilter $H(z)$ is the allpole filter given by

$$H(z) = \frac{6z}{1 + 4z^{-1} + z^{-2}}$$

The denominator is a symmetric polynomial, which shows that at least one pole has magnitude ≥ 1. Indeed the poles are -3.7321 and -0.2679. This shows that there is no **causal stable implementation**. Efficient noncausal implementations which make the spline interpolation very practical are described in [19]. The spline interpolation filter $\Phi_{d,L}(z)$ for $L = 2$ is given by

$$\Phi_{d,2}(z) = \frac{z^{-1}}{48} \left(1 + 8z^{-1} + 23z^{-2} + 32z^{-3} + 23z^{-4} + 8z^{-5} + z^{-6} \right)$$

Figure 12 shows a 128×128 portion of the Eve image and Fig. 13 shows the 2-fold interpolated version (256×256) obtained by using the above filters $H(z)$ and $\Phi_{d,2}(z)$ in Fig. 11.

[5]Note that the $\Phi(j\omega)$ decays as $1/\omega^{N+1}$. In some papers, this decay rate $(N+1)$ is regarded as the spline order.

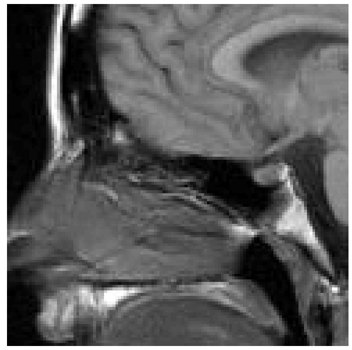

Figure 12. A 128×128 region of the Eve image.

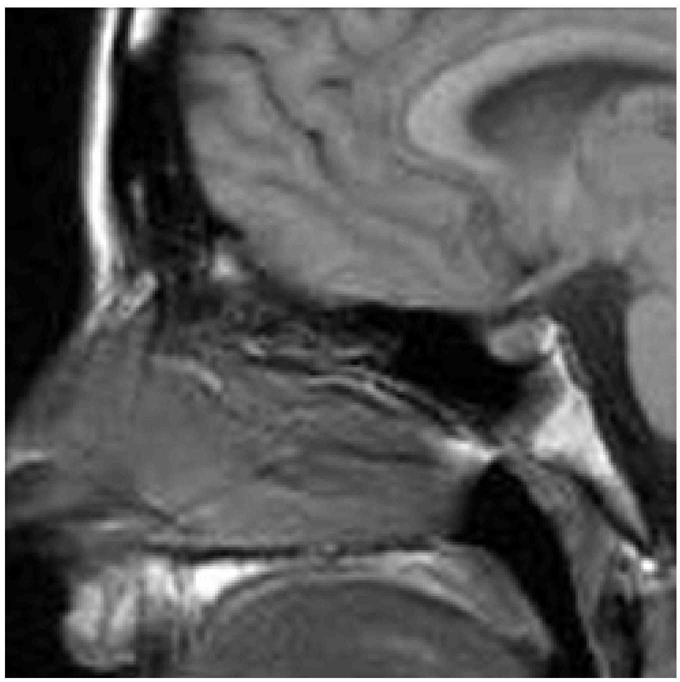

Figure 13. 256×256 interpolated version of the Eve image of Fig. 12, using the structure of Fig. 11 where $\phi(t)$ is the cubic spline.

6.2. All-FIR Interpolation

Consider again the signal model $x(t) = \sum_{k=-\infty}^{\infty} c(k)\phi(t-k)$ but assume that we are given the samples at the finer spacing $T = 1/2$. That is, we are given the **oversampled** version

$$x(n/2) = \sum_{k=-\infty}^{\infty} c(k)\phi\Big(\frac{n}{2} - k\Big)$$

In this case we can often find the interpolated samples $x(n/K)$ for any K using only FIR filters. To see this let $\phi_{d,2}(n) = \phi(n/2)$ as usual and rewrite the preceding as $x(n/2) = \sum_{k=-\infty}^{\infty} c(k)\phi_{d,2}(n-2k)$

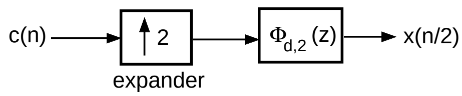

Figure 14. Interpreting $x(n/2)$ as the output of an interpolation filter $\Phi_{d,2}(z)$, where $\phi_{d,2}(n) = \phi(n/2)$.

This shows that $x(n/2)$ is the ouput of a digital interpolation filter as shown in Fig. 14. If $\Phi_{d,2}(z)$ is FIR and satisfies the conditions of Theorem 2 (with $M = 2$) then it has an FIR biorthogonal partner $H(z)$ to recover $c(n)$ from $x(n/2)$. Once we have $c(n)$ we can compute $x(t)$ for any t, for example, we can compute it at the finer spacing $1/K$ by observing that

$$x(n/K) = \sum_{k=-\infty}^{\infty} c(k)\phi\Big(\frac{n}{K} - k\Big) = \sum_{k=-\infty}^{\infty} c(k)\phi_{d,K}(n - Kk)$$

Thus we get the interpolation scheme shown in Fig. 15 where $H(z)$ and $\Phi_{d,K}(z)$ are both FIR.

Summary. Suppose a discrete time signal $x_d(n)$ can be modelled as $x_d(n) = x(n/2)$ where $x(t)$ is a continuous time signal modelled as $x(t) = \sum_k c(k)\phi(t-k)$. That is $x_d(n)$ is an oversampled version of $x_d(n)$ with an oversampling factor of two. Then we can recover the samples at finer spacings such as $x(n/K)$ by using the multirate system shown in Fig. 15. If $\phi(t)$ has finite duration then $\Phi_{d,K}(z)$ is FIR. If the two polyphase components of $\Phi_{d,2}(z)$ do not have common zeros then the filter $H(z)$ can be chosen to be FIR as well. Finally we would like to point out that it is possible to compute $c(n)$ using FIR filters even without oversampling of any kind. The trick is to use **nonuniform sampling**, as shown in [8].

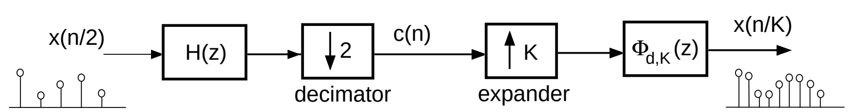

Figure 15. Interpolation of a signal $x(n/2)$ with digital filters. The signal is assumed to have a continuous time model $x(t) = \sum_k c(k)\phi(t-k)$. It is possible to make $H(z)$ FIR for finite-duration $\phi(t)$.

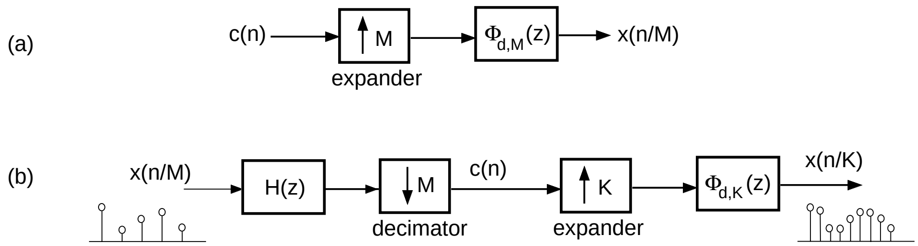

Figure 16. (a) Model for $x(n/M)$, and (b) further interpolation of $x(n/M)$ with digital filters. The underlying continuous time model is $x(t) = \sum_k c(k)\phi(t-k)$. It is possible to make $H(z)$ FIR for finite-duration $\phi(t)$.

Generalization

If $x_d(n) = x(n/M) = \sum_{k=-\infty}^{\infty} c(k)\phi(n/M - k)$ then we can represent it as in Fig. 16(a) where $\Phi_{d,M}(z)$ is a digital filter with impulse response $\phi_{d,M}(n) \stackrel{\Delta}{=} \phi(n/M)$. This is an FIR filter if $\phi(t)$ has finite duration. If there is no common zero shared by all the M polyphase components of $\Phi_{d,M}(z)$ then, according to Theorem 2, there exists an FIR filter $H(z)$ such that $c(n)$ can be recovered from $x(n/M)$ as shown in Fig. 16(b). Thus we can obtain interpolated versions $x(n/K)$ for any $K > M$ using the structure of Fig. 16(b). In fact we can even take $K < M$ which yields fractional decimation by M/K.

Example 3. For example assume again that $\phi(t)$ is the cubic spline. In this case

$$\Phi_{d,2}(z) = \frac{z^{-1}}{48}\Big(1 + 8z^{-1} + 23z^{-2} + 32z^{-3} + 23z^{-4} + 8z^{-5} + z^{-6}\Big)$$

This can be written in the polyphase form $\Phi_{d,2}(z) = R_0(z^2) + zR_1(z^2)$ where

$$R_0(z) = \frac{8z^{-1} + 32z^{-2} + 8z^{-3}}{48}, \quad R_1(z) = \frac{z^{-1} + 23z^{-2} + 23z^{-3} + z^{-4}}{48}$$

These polynomials are coprime. This can be verified either by running Euclid's algorithm or by explcit computation of their zeros (the finite zeros of $R_0(z)$ are -3.73 and -0.268 whereas those of $R_1(z)$ are $-21.95, -1$ and -0.0455 showing that these are coprime). So there exist FIR filters $E_0(z)$ and $E_1(z)$ such that $E_0(z)R_0(z) + E_1(z)R_1(z) = 1$. Indeed the pair

$$E_0(z) = \frac{3z^5 + 68z^4 + 47z^3}{6}, \quad E_1(z) = \frac{-24z^5 - 88z^4}{6}$$

yields $E_0(z)R_0(z) + E_1(z)R_1(z) = 1$. The FIR filter $H(z)$ in Fig. 15 is therefore

$$H(z) = E_0(z^2) + z^{-1}E_1(z^2) = \frac{3z^{10} - 24z^9 + 68z^8 - 88z^7 + 47z^6}{6}$$

To demonstrate, we consider the 256×256 oversampled image of Eve shown in Fig. 13. Then we can model it satisfactorily as $x(n/2) = \sum_{k=-\infty}^{\infty} c(k)\phi(n/2 - k)$ where $\phi(t)$ is the cubic spline. Suppose we want to interpolate this into a 512×512 image. Then we can do it using the scheme of Fig. 15 where $H(z)$ and

$\phi_{d,4}(z)$ are FIR filters. The result of interpolation is shown in Fig. 17. For comparison Fig. 18 shows the result of direct 4-fold interpolation of the 128×128 section using the standard non causal IIR filter method [19].

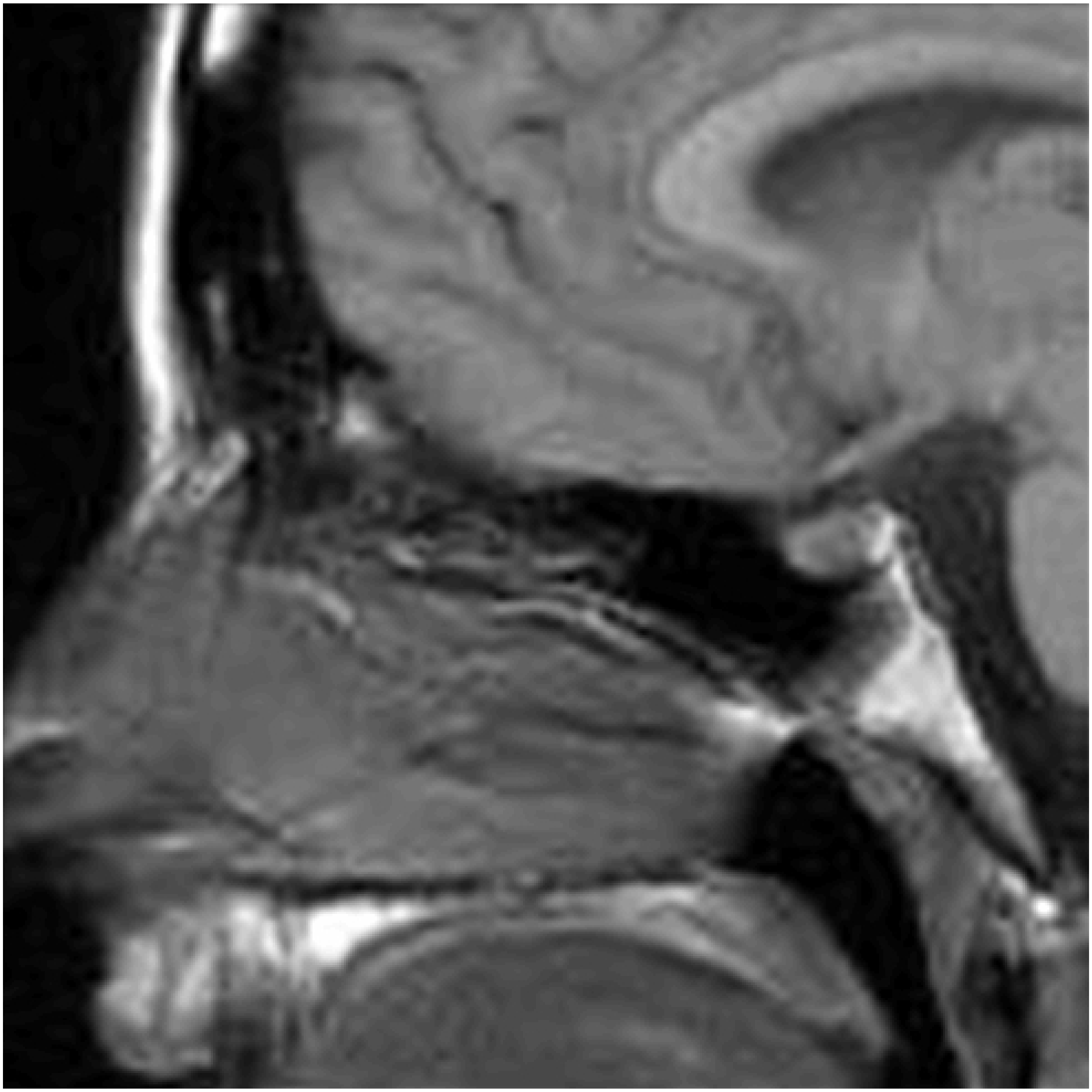

Figure 17. FIR based two-fold cubic-spline interpolation of the 256×256 Eve image shown in Fig. 13.

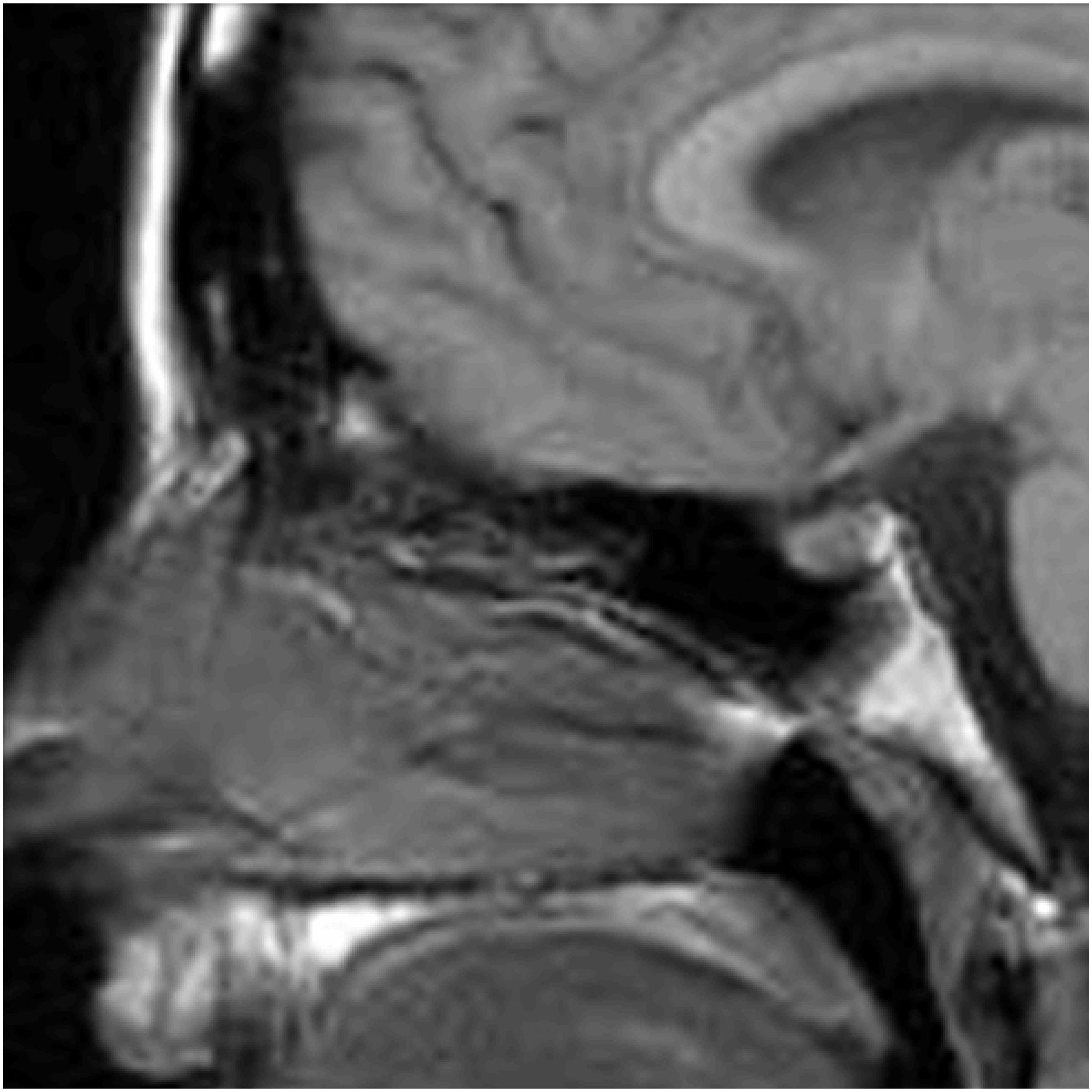

Figure 18. Direct four-fold cubic-spline interpolation of the 128×128 region of Eve image using traditional IIR method [19].

6.3. Application in Multiresolution Theory

For signals of the form $x(t) = \sum_k c(k)\phi(t-k)$ where $\phi(t)$ is a fixed finite duration function, if we only have the samples $x(n)$ then we need the IIR filter $1/\Phi_d(e^{j\omega})$ to compute $c(k)$. We also just showed that if the oversampled version $x(n/2)$ is available then we can compute $c(k)$ using only FIR filters. This is an attractive alternative to what is conventionally done in multiresolution analysis[6] to compute $c(k)$ from a highly oversampled version. To appreciate the difference between the above FIR construction and the conventional construction, we now give a brief review of the latter. Assuming $c(k)$ is in ℓ_2 and $\phi(t) \in L^2$, the set of functions $x(t) = \sum_k c(k)\phi(t-k)$ forms a subspace $V_0 \subset L^2$. This subspace is spanned by the integer-shifted versions $\{\phi(t-k)\}$ (Fig. 19(a)).

Now consider the squeezed version $\phi(2t)$ and its shifted versions $\phi(2t-k) = \phi(2(t-0.5k))$ sketched in Fig. 19(b). This set $\{\phi(2(t-0.5k))\}$ also spans a subspace $V_1 \subset L_2$. In multiresolution theory $\phi(t)$ is chosen such that $V_0 \subset V_1$. In particular $\phi(t)$ is a linear combination of $\phi(2t-k)$, that is,

$$\phi(t) = 2\sum_m f(m)\phi(2t-m) \tag{11}$$

This is the familiar **dilation equation** [6], [10] and translates in the Fourier domain to $\Phi(j\omega) = F(e^{j\omega/2})\Phi(j\omega/2)$. By repeating this idea we see that $x(t)$ belongs to the space V_J spanned by $\{\phi(2^J t-k)\}$ for any integer $J \geq 0$, that is,

$$x(t) = \sum_k c_J(k)2^{J/2}\phi(2^J t-k) \tag{12}$$

The multiresolution coefficients $c_J(k)$ at scale J reduce to the usual $c(k)$ for $J=0$. The constant $2^{J/2}$ merely ensures that the scaled basis functions $\{2^{J/2}\phi(2^J t-k)\}$ have the same energy for all J. Fig. 19(c) shows $x(t)$ and several shifted versions $\phi(2^J t-k)$. We shall now argue that the samples $x(2^{-J}k)$ are approximately proportional to $c_J(k)$. Since $x(2^{-J}n) = \sum_k c_J(k)2^{J/2}\phi(n-k)$, the sequence $x(2^{-J}n)$ is the output of the digital filter $\phi(n)$ in response to the input $2^{J/2}c_J(n)$. Thus, except for a constant multiplier, $c_J(n)$ is the output of the inverse filter $\gamma(n)$ in response to the input $x(2^{-J}n)$. If J is large enough, then $x(2^{-J}n)$ is nearly constant in the region where $\gamma(n)$ is significant. Thus the output $c_J(n)$ is also slowly varying, and is nearly proportional to the input, that is, $c_J(k) \approx \alpha_J x(2^{-J}k)$. If the oversampling factor 2^J is large enough, this estimate of $c_J(k)$ is very good. The beauty of the dilation equation is that it allows us to compute the multiresolution coefficients at lower scales

$$c_{J-1}(k),\ c_{J-2}(k),\ \ldots,\ c_0(k)$$

successively from $c_J(k)$, and thereby identify $c(k) = c_0(k)$. A brief justification of this well known result is given next for completeness.

[6]We refer the reader to Mallat's book [10] for an excellent treatment of multiresolution theory.

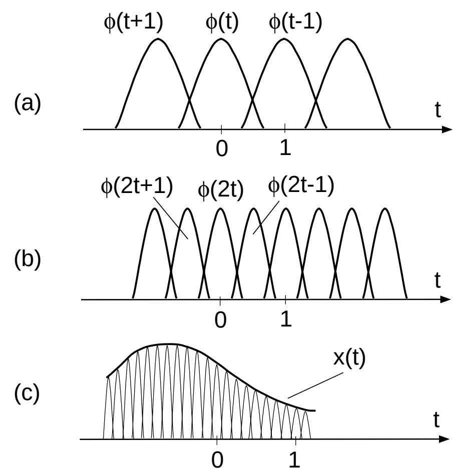

Figure 19. (a) The function $\phi(t)$ and its integer shifted versions, (b) the squeezed function $\phi(2t)$ and its shifted versions $\phi(2t-k)$, and (c) several shifted and weighted versions $c_J(k)2^{J/2}\phi(2^Jt-k)$ for $2^J=8$, shown along with $x(t)=\sum_k c_J(k)2^{J/2}\phi(2^Jt-k)$.

Proof. Substituting the dilation equation (11) into the scale-J representation (12), we get

$$\begin{aligned} x(t) &= \sqrt{2}\sum_m \sum_k c_J(k) f(m)\, 2^{(J+1)/2}\phi(2^{J+1}t-2k-m) \\ &= \sum_\ell \underbrace{\sum_k \sqrt{2} c_J(k) f(\ell-2k)}_{c_{J+1}(\ell)}\; 2^{(J+1)/2}\phi(2^{J+1}t-\ell) \end{aligned}$$

That is, $c_{J+1}(n)=\sqrt{2}\sum_k c_J(k)f(n-2k)$, which shows that we can go from scale J multiresolution coefficients $c_J(n)$ to the scale $J+1$ coefficients $c_{J+1}(n)$ by using an interpolation filter as shown in Fig. 20(a), where $F(z)=\sum_n f(n)z^{-n}$. If $\sqrt{2}F(z)$ has a **biorthogonal partner** $\sqrt{2}H(z)$ we can also go from scale $J+1$ to J by using the decimation filter of Fig. 20(b) (Sec. 2). This shows that we can compute the coefficients $c_i(n)$ for all lower scales using the **multistage decimation** system shown in Fig. 20(c). ▽ ▽ ▽

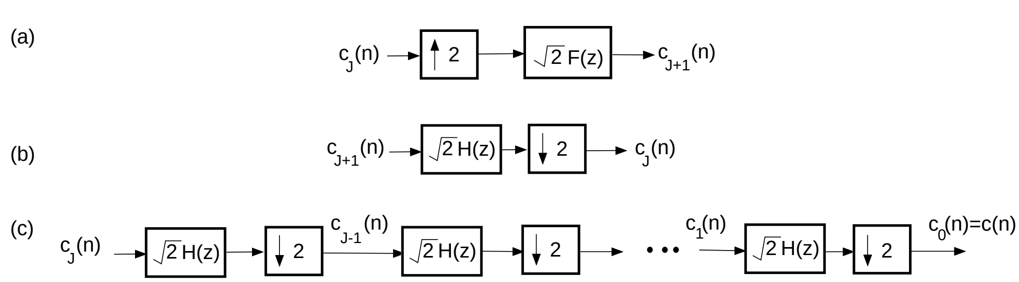

Figure 20. Details of conventional multiresolution computation. (a) Representation of $c_{J+1}(n)$, (b) computation of $c_J(n)$ from $c_{J+1}(n)$, where $\sqrt{2}H(z)$ is a biorthogonal partner of $\sqrt{2}F(z)$, and (c) multistage decimation circuit for computation of the coefficients $c_i(n)$ for all lower-level scales, upto $c_0(n)=c(n)$.

If $F(z)$ is FIR with coprime polyphase components $R_0(z)$ and $R_1(z)$ (where $F(z) = R_0(z^2) + zR_1(z^2)$) then we can find an FIR filter $H(z)$ to implement Fig. 20(c). Notice finally that if $\{\eta_k(n)\} = \{\sqrt{2}f(n-2k)\}$ is an orthonormal set then $h(n) = f^*(-n)$ works in the preceding scheme (Sec. 2). In order for the above oversampling strategy to yield good results, we have to make the oversampling factor large so that the approximation of $c_J(n)$ is good. Compare this with the method of Sec. 6.2 which yields exact results and requires oversampling only by a factor of two, and the method in [8] which yields exact results with no oversampling at all (but uses nonuniform sampling).

7. FRACTIONALLY SPACED EQUALIZERS

Consider the digital communication system shown in Fig. 21. Here $x(n)$ represents a sequence of symbols with spacing T seconds or **symbol rate** $1/T$ Hz. To be specific assume that $x(n)$ can have N possible amplitudes as in N-ary pulse amplitude modulation (PAM) [12]. The transmitting filter with impulse response $g_c(t)$ generates the continuous-time baseband signal $x_c(t) = \sum_n x(n)g_c(t-nT)$ which is then sent through the channel (subscript c is used for continous-time quantities, e.g., $x_c(t), X_c(j\omega)$, etc.). The modulation step which imposes a carrier is ignored, as it does not affect our discussions. Assume the channel can be modeled as a continuous time LTI system with frequency response $C_c(j\omega)$, followed by an additive noise source as shown in the figure. The received signal $r_c(t)$ is then passed through a receiver filter $h_c(t)$ and sampled at the symbol rate $1/T$. The resulting sequence $y(n)$ is input to the detector. Define

$$F_c(j\omega) \stackrel{\Delta}{=} G_c(j\omega)C_c(j\omega)$$

The purpose of the receiver filter or equalizer $h_c(t)$ is to make sure that $F_c(j\omega)H_c(j\omega)$ has the Nyquist(T) property (zero crossings at integer multiples of T) so that intersymbol interference or **ISI** is avoided.

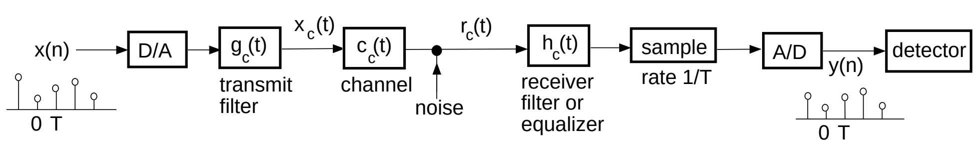

Figure 21. A digital communication channel.

In practice the filter $g_c(t)$ can be implemented digitally before D/A conversion into the channel, and the filter $h_c(t)$ implemented digitally after A/D conversion at the receiver. Assuming that the sampling at the receiver is done at the symbol rate $1/T$ we obtain a digital equivalent of the entire system as shown in Fig. 22(a). Here $G(z)$ is an M-fold interpolation filter (M a large enough integer so the pulse shape $g_c(t)$ is represented well by $g(n)$). The filter $C(z)$ represents the discrete time equivalent of $C_c(j\omega)$ sampled at M times the symbol rate $1/T$, and $H(z)$ is the digital filter representing the equalizer. Using the polyphase identity [23] this digital equivalent can be simplified as shown in Fig. 22(b) where $F(z) = [G(z)C(z)]_{\downarrow M}$. We say that $H(z)$ is the **symbol-spaced** equalizer or **SSE** because it operates at the symbol rate $1/T$. The discrete time equivalent of the noise source can readily be identified and is not the main point of the discussion here.

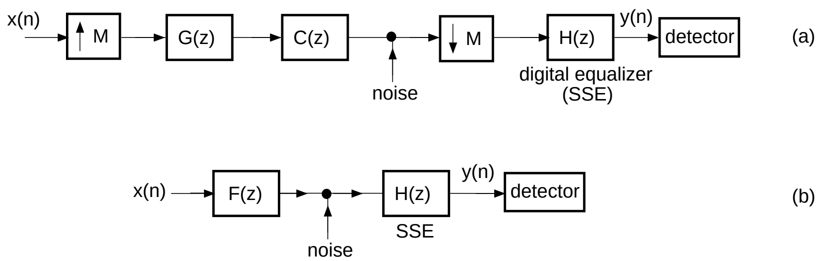

Figure 22. (a) All-digital equivalent of the digital communication channel, and (b) further simplification where $F(z) = [G(z)C(z)]_{\downarrow M}$.

The filter $F(z)$ is often represented well with an FIR approximation. An ideal equalizer (or a zero-forcing equalizer [12]) has the form $H(z) = 1/F(z)$ which is IIR, and typically of high order. In real time implementations the ideal equalizer is replaced with a practical FIR adaptive filter. It can be shown that the ISI suppression achieved by this filter is quite sensitive to the phase of sampling at the receiver [12]. The use of a so called **fractionally spaced** equalizer or **FSE** significantly reduces this problem and moreover allows FIR solutions [17].

To explain what an FSE is, consider again Fig. 21. Suppose the received signal is sampled at twice the symbol rate $2/T$. We then use an equalizer $H_2(z)$ and downsample its output by 2 before sending it to the detector. The system can then be represented in discrete time multirate notation by Fig. 23(a) (assuming M is even). This can be simplified into the form shown in Fig. 23(b) where $F_2(z) = [G(z)C(z)]_{\downarrow M/2}$. (Again the noise source can be adjusted accordingly). We see that the effective transfer function between the transmitted symbols $x(n)$ and received symbols is now $[F_2(z)H_2(z)]_{\downarrow 2}$. This can be made unity by designing $H_2(z)$ to be a **biorthogonal partner** of $F_2(z)$. The filter $H_2(z)$ is the fractionally spaced equalizer. It operates at twice the symbol rate. Typically $F_2(z)$ represents an FIR approximation of $G_c(j\omega)C_c(j\omega)$. Thus, according to Theorem 2 it is almost always possible to find an **FIR equalizer** $H_2(z)$, the only mild condition being that the two polyphase components of $F_2(z)$ be coprime. The FSE technique not only offers an FIR solution, it also reduces significantly the sensitivity of the equalizer performance to errors in sampling phase as demonstrated amply in [12].

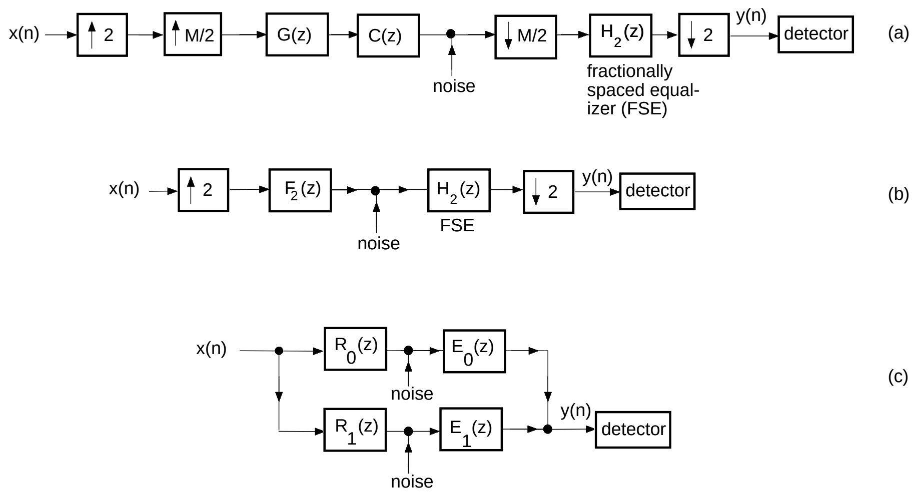

Figure 23. The idea of a fractionally spaced equalizer (FSE). (a) Equalizer $H_2(z)$ operating at the rate $2/T$, (b) further simplification where $F_2(z) = [G(z)C(z)]_{\downarrow M/2}$, and (c) the polyphase representation.

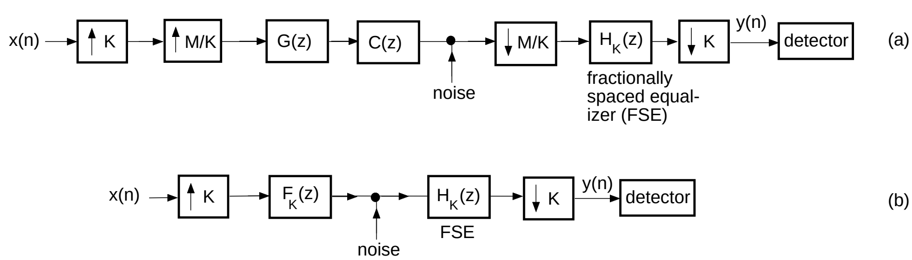

Figure 24. Generalization of the fractionally spaced equalizer (FSE). (a) Equalizer $H_K(z)$ operating at the rate K/T, and (b) further simplification where $F_K(z) = [G(z)C(z)]_{\downarrow M/K}$.

Using the polyphase decompositions $H_2(z) = E_0(z^2) + z^{-1}E_1(z^2)$ and $F_2(z) = R_0(z^2) + zR_1(z^2)$ we can redraw the structure of Fig. 23(b) as in Fig. 23(c) as shown in the proof of Theorem 2. Here $R_0(z)$ represents the even samples of the impulse response of $F_2(z)$ and $R_1(z)$ represents the odd samples. Notice that $R_0(z)$ is precisely the quantity $F(z)$ in the symbol-spaced equalizer scheme of Fig. 22(b). Thus the FSE structure takes into account the two interleaved sets of samples $R_0(z)$ and $R_1(z)$ from the channel, rather than just the samples from $R_0(z)$. This is the secret of its improved performance: the FSE "reads between lines", i.e., takes into account the samples of the effective channel between symbols. An obvious extension of the FSE scheme given above would be to use a finer spacing, e.g., K times the symbol rate. Then Fig. 23(a) and 23(b) would be modified as shown in Fig. 24 where $F_K(z) = G(z)C(z)_{\downarrow M/K}$ (assuming K is a factor of M).

Many papers have been written on the topic of FSE. An excellent tutorial is the paper by Treichler et al. [17]. Sec. 10-2-4 of [12] is an insightful exposure, and cites pioneering references. The paper by Tong et al. [15, 16] develops many theoretical results some of which are applicable to the above context.

8. CONCLUDING REMARKS

Transfer function pairs called biorthogonal partners arise naturally in many signal processing applications. In this paper we have provided a unified treatment of biorthogonal partners with some new results, and taken a second look at some of the well known results. Several applications were pointed out. This treatment has been possible because of contributions from many researchers in the filter bank and wavelet areas.

References

[1] Akansu, A. N., and Haddad, R. A. *Multiresolution signal decomposition,* Academic Press Inc., 1992.
[2] Aldroubi, A. "Oblique projections in atomic spaces", Proc. Amer. Math. Soc., vol. 124, pp. 2051–2060, 1996.
[3] Aldroubi, A., and Unser, M. "Oblique projections in discrete signal subspaces of ℓ_2 and the wavelet transform", Proc. SPIE. vol. 2303, *Wavelet applications in signal and image processing, II*, pp. 36–45, San Diego, CA, 1994.
[4] Cohen, A., and Daubechies, I. "A stability criterion for biorthogonal wavelet bases and their related subband coding scheme," *Duke Mathematical Journal*, vol. 68, no. 2, pp. 313–335, 1992.
[5] Cohen, A., Daubechies, I., and Feauveau, "Biorthogonal bases of compactly supported wavelets," *Comm. Pure Appl. Math.*, vol. 45, no. 2, pp. 485–560, 1992.
[6] Daubechies, I. *Ten lectures on wavelets,* SIAM, CBMS series, April 1992.
[7] de Boor, C. *A Practical Guide to Splines*, New York: Springer-Verlag, 1978.
[8] Djokovic, I., and P. P. Vaidyanathan, "Generalized sampling theorems in multiresolution subspaces", IEEE Trans. Signal Proc., vol. SP-45, pp. 583–599, Mar. 1997.
[9] Mallat, S. "A theory for multiresolution signal decomposition: the wavelet representation," IEEE Trans. PAMI, pp. 674-693, July 1989.
[10] Mallat, S. *A wavelet tour of signal processing*, Academic Press, 1998.
[11] Oppenheim, A. V., Willsky, A. S., and Young, I. T. *Signals and systems,* Prentice Hall, Inc., 1983.
[12] Proakis, J. G. *Digital communications,* McGraw Hill, Inc., 1995.
[13] Schoenberg, I. J. *Cardinal spline interpolation* Siam, 1973.
[14] Strintzis, M. G. "Optimal pyramidal and subband decompositions for hierarchical coding of noisy and quantized images" IEEE Trans. Image Proc., pp. 155–166, Feb. 1998.
[15] Tong, L, Xu, G., and Kailath, T. "Blind identification and equalization based on second order statistics: a time domain approach" IEEE Trans. Info. Theory, pp. 340–349, March 1994.
[16] Tong, L, Xu, G., and Kailath, T. "Blind identification and equalization based on second order statistics: a frequency domain approach" IEEE Trans. Info. Theory, pp. 329–334, Jan. 1995.
[17] Treichler, J. R., Fijalkow, I., and Johnson, C. R., Jr., "Fractionally spaced equalizers: how long should they be?" IEEE Signal Processing Magazine, pp. 65–81, May 1996.
[18] Unser, M. and Aldroubi, A. "Polynomial splines and wavelets", in *Wavelets, a tutorial in theory and applications,* ed., C. K. Chui, Academic Press, 1992.
[19] Unser, M., Aldroubi, A., and Eden, M. "Fast B-spline transforms for continuous image representation and interpolation" IEEE Trans. PAMI, pp. 277-285, March 1991.
[20] Unser, M., Aldroubi, A., and Eden, M. "B-spline signal processing: Part I — Theory", IEEE Trans. Signal Proc., vol. 41, no. 2, pp. 821–833 , Feb. 1993.
[21] Unser, M., Aldroubi, A., and Eden, M. "B-spline signal processing: Part II — Efficient design and applications", IEEE Trans. Signal Proc., vol. 41, no. 2, pp. 834–848 , Feb. 1993.
[22] Unser, M. "Sampling — 50 years after Shannon", Proc. of the IEEE, vol. 88, no. 4, pp. 569–587, April 2000.
[23] Vaidyanathan, P. P. *Multirate systems and filter banks,* Prentice Hall, Inc., 1993.
[24] Vaidyanathan P. P. and Djokovic, I. "Wavelet transforms," in *The Circuits and Filters Handbook,* edited by W. K. Chen, CRC Press, Inc., pp. 134–219, 1995.
[25] Vaidyanathan, P. P. and Phoong, S-M. "Reconstruction of sequences from nonuniform samples," Proc. IEEE Int. Symp. Circuits and Sys., Seattle, Washington, April-May, 1995.
[26] Vaidyanathan, P. P. and Phoong, S-M. "Discrete time signals which can be recovered from samples," Proc. IEEE Int. Conf. Acoust. Speech, and Signal Proc., Detroit, May 1995.
[27] Vetterli, M., and Kovačević, J. *Wavelets and subband coding,* Prentice Hall, Inc., 1995.
[28] Walter, G. G. "A sampling theorem for wavelet subspaces," IEEE Trans. IT, pp. 881-884, March 1992.

FOOTNOTES

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