# OPTIMALITY OF ORTHONORMAL TRANSFORMS FOR SUBBAND CODING

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## ABSTRACT

Theory of optimal orthonormal filter banks has recently been developed under the assumption that quantizers operate at high bit rates. We show that the theory can be simplified if a more general model for quantizers is used. With such a model, we show that principal component filter banks (PCFB) are optimal for orthonormal subband coding under all bit rates and bit allocation strategies. This establishes the link between two fundamental problems: principal component representation of signals and optimal orthonormal subband coding. In block transform case, the Karhunen-Loeve transform is a PCFB. At the other extreme, PCFB's with ideal filters have recently been constructed. In the intermediate case of FIR orthonormal filter banks, it has recently been shown that there does not always exist a PCFB for a given input. This case is addressed in this paper with some new insights.

## 1. INTRODUCTION

Optimization of filter banks for subband coding (SBC) of signals has been an active area of research [1, 2, 10, 17, 18]. Subband coding involves a linear transform part and a nonlinear quantization part. Block transform coding, overlapped transform coding, and wavelet coding are special cases.

In the block transform coding case, the optimal orthogonal transform matrix is well known to be the Karhunen Loeve transform. To show the optimality of KLT, one usually resorts to a quantization model that is valid only at high bit rates. However, a classic work by Huang and Schultheiss [3] contains a proof that KLT is optimal even if the quantizers can not be assumed to have high bit rates. Inspired by this pioneering work, we provide a very simple proof of optimality of orthonormal filter banks under a general quantizer noise model without assuming high bit rates. Our simplified point of view enables us to state a very strong result that principal component filter banks are optimal for subband coding for all bit rates and bit allocation strategies. We have been able to show that the problems of optimal representation of signals using principal component analysis and optimal lossless subband coding are fundamentally the same. It should be noted however that, in general, biorthogonal filter banks perform better than orthonormal filter banks in subband coding [18]. The only exception to this is the case of block transform coding where the KLT is orthonormal and no other biorthogonal transform can result in a better coding performance [17].

The paper is organized as follows: In Sec. 2, we briefly review the optimality of orthonormal filter banks under the classical high bit rate quantizer models. Sec. 3 contains our simplified proof under more general quantizer models without assuming high bit rates. In this section, we also establish the link between the problems of optimal orthonormal subband coding and principal component representation of signals. Sec. 4 is concerned with the case where the filters are constrained to be FIR. All the results in this paper pertain to uniform orthonormal filter banks. Theoretical optimization of nonuniform filter banks for subband coding has proven to be very difficult. Utilizing the simplified point of view of this paper, we are now able to obtain important results in the nonuniform case as well. A detailed presentation of results in the nonuniform case is a topic of another work [5].

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## 2. OPTIMAL SBC WITH HIGH BIT RATE QUANTIZERS

Consider Fig. 1, where we show a subband coding scheme with a uniform filter bank as its transform part. Fig. 2 shows the polyphase representation for the analysis part [16]. An orthonormal filter bank is one with a paraunitary $E(z)$ which means that $E(e^{j\omega})$ is unitary for each $\omega$. In the special block transform coding case, we have $E(z) = E$, a constant matrix, and orthonormality is the same as $E$ being unitary. Let $\sigma_x^2$ be the original signal variance, $\sigma_{x_j}^2$ be the subband variances and $\sigma_{q_j}^2$ be the quantization noise variances in the subbands. If we assume that the quantizers operate at high bit rates, then one can model them by a simple relationship: $\sigma_{q_j}^2 = c2^{-2b_j}\sigma_{x_j}^2$, where $b_j$ is the number of bits at which the quantizer in $j$th channel is operating. If the total bit budget is $b$ bits per pixel, then we have $b = \sum_{j=1}^{M} b_j/M$. Coding gain of a SBC scheme is defined to be the ratio of the quantization error when the input is quantized directly and the reconstruction error of the SBC scheme using the same bit budget $b$. In the high bit rate case, since we have analytical expressions for quantizer variances, we can optimally allocate the bits among the channels and finally obtain the following formula for coding gain [4] :

$$G_{coding} = \frac{\sigma_x^2}{(\prod_{j=1}^{M}\sigma_{x_j}^2)^{1/M}} \tag{1}$$

We have $\sigma_x^2 = (\sum_{j=1}^{M}\sigma_{x_j}^2)/M$ by orthonormality of the filter bank. Hence the coding gain in the high bit rate case is the ratio of arithmetic and geometric means of subband variances. In the transform coding case, by a well known inequality in linear algebra, this is maximized if and only if the transform is the KLT which diagonalizes the autocorrelation matrix of the input.

Block transform coding with $E(z) = E$, corresponds to subband coding with filters having lengths equal to the number of channels. Recently optimal orthonormal filter banks for subband coding with ideal filters have been constructed in [17]. The construction is proven to be optimal under the assumption that the quantizers operate at high bit rates and therefore the coding gain expression (1) is valid. The optimal paraunitary polyphase matrix $E(e^{j\omega})$ is shown to have two properties [17] :

i) it diagonalizes the power spectrum matrix of the input vector at each frequency (total decorrelation),

ii) its rows at each frequency are ordered in such a way that the corresponding diagonal elements are ordered as well.

Loosely speaking, optimal $E(e^{j\omega})$ for a given $\omega$ is the Karhunen Loeve transform matrix corresponding to the input power spectrum matrix evaluated at the frequency $\omega$. In the biorthogonal case, it is shown in [18] that the above orthonormal solution can be improved by further processing of subband signals. This final stage consists of half-whitening of uncorrelated subband signals.

## 3. A SIMPLIFIED PROOF UNDER A MORE GENERAL QUANTIZER MODEL

A characterizing property of an orthonormal transform is that it preserves the total energy of its input. Hence orthonormal transforms are also referred as lossless transforms. This is the major property we use to prove the optimality of an orthonormal filter bank for subband coding. Consider Fig. 1 again. Let $\sigma_x^2, \sigma_{x_j}^2$, and $\sigma_{q_j}^2$ be variances of the original input, subband signals, and the quantization noise signals as defined before. Let $\mathcal{E}$ denote the reconstruction error of this subband coding scheme. Then, by the orthonormality of the transform we have : $\mathcal{E} = \sum_{j=1}^{M} \sigma_{q_j}^2 / M$. A filter bank is said to be optimum for subband coding if the reconstruction error $\mathcal{E}$ is minimized for a given bit budget, $b$ bits per input sample. At high bit rates, optimality is the same as maximization of the coding gain expression in (1). In general however, (1) does not represent the objective.

### 3.1. A more general quantizer model

We will model all the quantizers in Fig. 1 by a single quantization function $f(.)$ such that

$$\sigma_{q_j}^2 = f(b_j)\sigma_{x_j}^2 \qquad (2)$$

The important benefit of this model is that **we do not assume high bit rates**. However, we do assume that all the channels have the same quantizer function $f(.)$, which in general need not be true. In general, the quantizer functions $f_j(.)$ depend on the statistics of the $j$th channel signal, which in turn depend on the linear transformation. If, however, the original input signal is Gaussian, then regardless of which transform is used, the subband channels are all Gaussian as well, and hence can be modeled by a single quantizer function $f(.)$. We do not assume that the noise signals are white or uncorrelated with each other. The traditional high bit rate quantizer model can now be considered as a special case of our model with $f(b_j) = c2^{-2b_j}$. With our quantizer model, the reconstruction error of the subband coding scheme is

$$\mathcal{E} = \sum_{j=1}^{M} f(b_j)\sigma_{x_j}^2 / M \qquad (3)$$

### 3.2. Principal component representation of signals

Let us consider for a moment, a different but a closely related problem that deals with multiresolution representation of signals [13]. Consider Fig. 1 and let us not constrain the filter bank to be orthonormal. If we keep the first $P < M$ of the subband signals without quantizing, and if we drop the other subbands, there will be a corresponding reconstruction error. Intuitively, the minimization of this error is the same as putting most of the signal information into the first $P$ channels. If there is a single filter bank that minimizes this error for each $P = 1, \ldots, M$, we call it a **principal component filter bank**. It provides us an optimal multiresolution representation of the signal as $P$ increases from 1 to $M$. For $P = M$, the reconstruction error is zero if the filter bank is perfect reconstruction (that is, biorthogonal). It turns out that, if we allow ideal filters, there always exists a principal component filter bank [13] for any given input process, and it is orthonormal. Furthermore it is the same filter bank as the optimal orthonormal filter bank for subband coding as derived in [17] under high bit rate quantizer models. Is this a coincidence? The relationship between the two problems are clarified in the next section under our more general modeling of quantizers.

### 3.3. Principal component representation and optimal SBC

Consider Fig. 1 again and let the first $P < M$ of the subbands are kept. If the filter bank is orthonormal, then the reconstruction error is the sum of variances of dropped subbands, that is:

$$\mathcal{E} = \sum_{j=P+1}^{M} \sigma_{x_j}^2 / M \qquad (4)$$

Notice the similarity between (3) and (4). The latter can be seen as a special case of the former when $f(b_j) = 0,\ j = 1, \ldots, P$ (infinite precision) and $f(b_j) = 1,\ j = P+1, \ldots, M$ (zero bit assignment). By the lossless property, the sum of all subband variances is constant as noted before. Hence the error in (4) is minimized if $\sum_{j=1}^{P} \sigma_{x_j}^2 / M$ is maximized. A principal component filter bank as defined in the previous section maximizes the partial sum $\sum_{j=1}^{P} \sigma_{x_j}^2 / M$ for each $P$. We are now ready to state and prove our main result :

**Theorem.** *With a general quantizer model described in Sec. 3.1, a principal component filter bank, if it exists, minimizes the reconstruction error of an orthonormal subband coder for all bit budgets and bit allocation strategies.*

**Proof.** Without loss of generality assume that $f(b_1) \leq \ldots \leq f(b_M)$. By a simple algebra, we can write the error in (3) as

$$\mathcal{E} = \sum_{j=1}^{M-1} (f(b_j) - f(b_{j+1})) \sum_{i=1}^{j} \sigma_{x_i}^2 / M + f(b_M) \sum_{i=1}^{M} \sigma_{x_i}^2 / M \qquad (5)$$

By assumption, $f(b_j) - f(b_{j+1}) \leq 0$. The last term is fixed and equal to $f(b_M)\sigma_x^2$. Hence $\mathcal{E}$ is minimized if the partial sums $\sum_{i=1}^{j} \sigma_{x_i}^2$ are maximized for each $j$. This is the case if the filter bank is a principal component filter bank (PCFB). ∎

**Remarks.**

**1.** Notice that optimal bit allocation did not enter the discussion. A PCFB minimizes $\mathcal{E}$ for any bit allocation, in particular, for the optimal bit allocation. There are many interesting results on the optimum allocation of bits. We refer to the authorative work [12] for a general treatment of the topic. As long as we can model the quantizers with one underlying quantization function, the optimization of the orthonormal filter bank is decoupled from the optimization of bit allocation in subbands. Even if the bit allocation is not optimal, a PCFB continues to be optimal.

**2.** From the definition, the subband variances of a PCFB are in decreasing order: $\sigma_{x_1}^2 \geq \sigma_{x_2}^2 \geq \ldots \geq \sigma_{x_M}^2$. This is because, if this is not true, then one can get a new filter bank by reordering the filters such that the variances are ordered. The new filter bank will have a higher sum $\sum_{i=j}^{P} \sigma_{x_j}^2$ for at least one $P$ contradicting that the original filter bank was a PCFB.

**3.** Given the set of ordered variances $\sigma_{x_1}^2 \geq \sigma_{x_2}^2 \geq \ldots \geq \sigma_{x_M}^2$ of a PCFB, one can do optimal bit allocation using the quantizer function $f(.)$. Let $b_j^*, j = 1, \ldots, M$ denote this optimum allocation. If the inequalities between subband variances are strict, i.e., $\sigma_{x_1}^2 > \sigma_{x_2}^2 > \ldots > \sigma_{x_M}^2$, then we must have $f(b_1^*) \leq \ldots \leq f(b_M^*)$. To see this, assume $f(b_i^*) > f(b_{i+1}^*)$ for some $i$. Then by interchanging $b_i^*$ and $b_{i+1}^*$, we see that the total error $\mathcal{E} = \sum_{j=1}^{M} f(b_j^*)\sigma_{x_j}^2/M$ is reduced while maintaining the total bit budget, contradicting the optimality of the bit allocation. If however, for any $i$, $\sigma_{x_i}^2 = \sigma_{x_{i+1}}^2$, then one can interchange the bits to guarantee $f(b_i^*) \leq f(b_{i+1}^*)$ without affecting the total error.

**4.** As an artificial special case, let $P < M$ be fixed and let $f(b_j) = 0,\ j = 1, \ldots, P$, and $f(b_j) = 1,\ j = P+1, \ldots, M$. Then we have to maximize $\sum_{j=1}^{P} \sigma_{x_j}^2$ for that $P$. If we repeat the process for each $P = 1, \ldots, M-1$, and if there exists one single solution for all of them, then the solution is a PCFB!

**5.** As another special case, consider $f(b_j) = c$, a constant, for all $j$. This corresponds to equal bit allocation strategy. In this case we have, $\mathcal{E} = c\ \sigma_x^2$, which is independent of the transformation. Hence, orthonormal filter banks do not yield any coding advantage if equal bit allocation strategy is used.

**6.** Finally consider the case $f(b_1) < f(b_2) < \ldots < f(b_M)$. In this case, all $f(b_j) - f(b_{j+1}) < 0$, and therefore the partial sums $\sum_{i=0}^{j} \sigma_{x_i}^2$ must be maximized for all $j$. Hence a PCFB, if exists, is not only sufficient but also necessary for optimality.

#### *3.3.1. Underlying mathematical theory*

The result that we have just presented is yet another application of a topic in applied mathematics called **majorization** [9]. A set of $M$ numbers $a_j,\ j = 1, \ldots, M$ is said to majorize another set of $M$ numbers $b_j,\ j = 1, \ldots, M$, if $\sum_{j=1}^{P} a_j \geq \sum_{j=1}^{P} b_j,\ P = 1, \ldots, M$, with equality if $P = M$. We see that a PCFB is the one that produces a set of subband variances that majorizes sets of subband variances obtained by all other filter banks. There are many other applications of majorization. The interested reader is referred to [9] for a beautiful treatment of the subject and to [11] for more recent applications.

## 4. FIR ORTHONORMAL FILTER BANKS

The result developed in previous sections asserts that for a given input, whenever one filter bank produces a set of subband variances that majorizes sets of subband variances of all other filter banks, it is optimal for subband coding for that input under all bit rates and bit allocation strategies. We have not put any constraints on the filters. Now, we can think of a class of orthonormal filter banks that is characterized by practical constraints such as finite filter lengths. The simplest case is the block transforming case we have considered at the beginning. In this case, the autocorrelation matrices of subband and original input signals are related via a unitary matrix. It is a well known fact in linear algebra that the eigenvalues of an autocorrelation matrix majorizes its diagonal elements. This gives us the simplified proof that KLT is optimal among the class of block transforms. This is because, with KLT, eigenvalues become the subband variances. As before, the optimality is independent of the bit rates involved and bit allocation strategies.

So, in the two extreme cases where (1) filters can be ideal and (2) filters are constrained to be of the same length as the number of channels, we know that a PCFB exists and therefore optimal for subband coding. What happens in the intermediate case where the filter lengths are finite but larger than the number of channels? This case turns out to be very difficult to analyze as confirmed by several researchers [10, 15, 19] who devised numerical techniques for suboptimal solutions. One approach for a suboptimal solution is to design an optimal energy compaction filter [7] that pushes most of the signal energy into the first channel, and then complete the filter bank in some optimal fashion [10]. Although this technique was claimed to find optimal solution in [10], we have shown that this is not the case in [6]. The reason for its suboptimality is that it is a loss of generality to design the first filter to be a compaction filter, as there may not even exist a PCFB! Another important approach is to develop structures that are efficient to implement [8] while hopefully not being too much away from the optimal solution. At present, there is no robust algorithm that converges to an optimal FIR orthonormal filter bank. A good strategy for large filter lengths is to try to approximate the optimal ideal filter banks.

### 4.1. On the existence of FIR PCFB's

The difficulty in FIR case starts with the exact definition of the class of filter banks in which an optimal solution is searched for. There are two equally plausible choices: those with a finite degree and those with a finite order. A brief description of the differences between the two is described next. For more details, the reader is referred to [6]. Let $E(z) = \sum_{n=0}^{K} E_n z^{-n}$, with $E_K \neq 0$. Then the order of $E(z)$ is said to be $K$. Note that filter lengths can be as high as $M(K+1)$. The notion of degree is different from the notion of order and it is defined to be the minimum number of delay elements to implement $E(z)$. If $\mu$ denotes degree, then in general, $\mu \geq K$. Notice that this ambiguity does not arise in block transform and ideal filters cases. In the former, both degree and order are zero, while in the latter they are infinite or undefined.

We have shown in [6] that an FIR PCFB of a given degree $\mu$ does not always exist. The following example shows that an FIR PCFB of a given order $K$ does not always exist either. These results help us understand why it is difficult to find analytical solutions for optimal FIR orthonormal filter banks.

**Example.** Let $M = 3$, and $K = 1$ (LOT case). If a PCFB exists, then it solves the following two problems : maximize $\sigma_{x_1}^2$, and maximize $\sigma_{x_1}^2 + \sigma_{x_2}^2$. The second problem is the same as the minimization problem : minimize $\sigma_{x_3}^2$. This is because, by orthonormality, $\sigma_{x_1}^2 + \sigma_{x_2}^2 + \sigma_{x_3}^2 = 3\sigma_x^2$. Both two problems can be seen as the problem of designing optimal FIR compaction filters

[7, 14]. We have used the numerical technique developed in [14] which is guaranteed to converge to optimal solutions. The solution to the maximization problem consists of a set of filter coefficients for the first filter $H_1(z)$. These correspond to all possible spectral factors of a single magnitude-squared response. Similarly, the minimization of $\sigma^2_{x_3}$ leads to a set of coefficients for the third filter $H_3(z)$. If it turns out that no combination of filters from the two sets of solutions can possibly belong to a single orthonormal filter bank, then this is a proof that a PCFB of order $K = 1$ does not exist. We next show that this is the case.

We can uniquely factorize the polyphase vector of each solution $H_1(z)$ into Householder factors: $\mathbf{e}_1^\dagger(z) = \mathbf{u}_1^\dagger V_1(z)V_2(z)$. Similarly for each $H_3(z)$ we have $\mathbf{e}_3^\dagger(z) = \mathbf{u}_3^\dagger W_1(z)W_2(z)$. The Householder factors are in the form $V(z) = I - \mathbf{v}\mathbf{v}^\dagger + z^{-1}\mathbf{v}\mathbf{v}^\dagger$ so that $V(1) = I$. Hence $\mathbf{e}_1^\dagger(1) = \mathbf{u}_1^\dagger$ and $\mathbf{e}_3^\dagger(1) = \mathbf{u}_3^\dagger$. If there was a single orthonormal filter bank, we would have had $\mathbf{u}_1^\dagger\mathbf{u}_2 = 0$ for at least one combination of solutions. For an AR(1) process with $\rho = 0.9$ we have explicitly found that this was not the case. Hence we conclude that, for this example, there does not exist a PCFB of order $K = 1$.

### 4.2. The two-channel case

In the special two-channel case, a PCFB is the one that maximizes $\sigma^2_{x_1}$. Hence an FIR PCFB can always be constructed by designing the first filter to be an optimal FIR compaction filter. The second filter is determined from the first. Hence, in the two-channel case, an FIR PCFB always exist for any input and it is optimal for orthonormal subband coding for all bit rates and bit allocation strategies. Furthermore, they can be analytically constructed for some special class of input signals [7]

## 5. CONCLUSION

We have clarified the relation between principal component filter banks and optimal orthonormal filter banks for subband coding. We have considered a general quantizer model which does not assume high bit rates. With such a model, we proved that principal component filter banks are optimal for orthonormal subband coders under all bit rates and bit allocation strategies. For a given input, a PCFB among a class of orthonormal filter banks is defined to be the one that has the largest partial sums of subband variances. In other words, the set of subband variances of a PCFB majorizes sets of subband variances of all other orthonormal filter banks. If the class of orthonormal filter banks is the unitary block transforms or ideal orthonormal filter banks, then a PCFB for any given input exists. We have shown that this is not the case if the class is FIR orthonormal filter banks of a given order.

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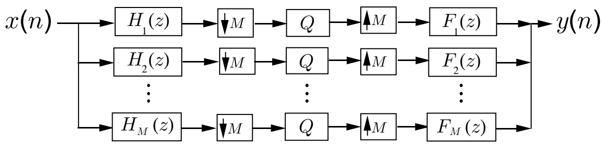


Figure 1: Subband coding scheme with a uniform filter bank

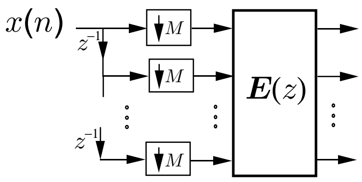


Figure 2: Polyphase representation of the analysis part