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ARE NONUNIFORM PRINCIPAL COMPONENT FILTER BANKS OPTIMAL?

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ABSTRACT[1]

The notion of a principal component filter bank (PCFB) for a given class of uniform filter banks (FB's) has been well studied. Recent work by the authors has shown that PCFB's are optimal orthonormal FB's whenever the minimization objective is a concave function of the vector of subband variances of the FB. This result gives a unified explanation of PCFB optimality for progressive transmission, compression, noise suppression, and as shown more recently, for use in DMT (discrete multitone modulation) systems. This paper generalizes such results to nonuniform FB's. We propose two distinct definitions of nonuniform PCFB's. Each definition results in PCFB optimality for certain types of concave objectives whose form is somewhat more restricted than in the case of uniform FB's. We study existence of the defined PCFB's, and observe that it can be very delicate: Small perturbations of the input spectra can sometimes destroy the existence of nonuniform PCFB's.

1 INTRODUCTION: UNIFORM PCFB's

Figure 1 shows a general subband signal processing scheme using a uniform M channel filter bank (FB). We only consider orthonormal FB's, i.e. where $F_k(e^{j\omega}) = H_k^*(e^{j\omega})$ and the FB obeys perfect reconstruction (PR) $y(n) = x(n)$ *in absence of subband processing.* The subband processors P_i are aimed at producing a certain desired signal $d(n)$ at the output. For example in data compression, the P_i are quantizers and $d(n) = x(n)$. In noise suppression, $x(n) = s(n) + \mu(n)$ where $\mu(n)$ is the additive noise, and $d(n) = s(n)$, the pure signal.

1.1 Formulating the FB optimization problem

Let $\mathcal{C}$ be a class of uniform orthonormal FB's. FB optimization involves picking the *best* FB in $\mathcal{C}$ for the given application and input statistics. By 'best FB' we mean one minimizing some well defined objective measure of the error $e(n) = y(n) - d(n)$ between the true and desired FB outputs in Fig. 1. It is common to use statistical models of the input by wide sense stationary (WSS) random processes, and to measure the error process $e(n)$ by its mean square value. Denote by $v_i^{(s)}(n)$ the i-th subband signal when the FB input is $s(n)$. Let the subband error processes $v_i^{(e)}(n) = v_i^{(y)}(n) - v_i^{(d)}(n)$ be jointly WSS, and let the (mean square) energy of $v_i^{(e)}(n)$ be

$$E[|v_i^{(e)}(n)|^2] = h_i(\sigma_i^2) \qquad (1)$$

where function h_i depends purely on the kind of subband processor P_i and is independent of choice of FB. Here $\sigma_i^2 = E[|v_i^{(x)}(n)|^2]$ is the i-th subband variance. (For facility we will often use the term 'variance' here, though for nonzero mean signals 'energy' is more correct.)

Under the above assumptions, the objective function to be minimized has the form

$$g(\sigma_0^2, \sigma_1^2, \ldots, \sigma_{M-1}^2) = \frac{1}{M}\sum_{i=0}^{M-1} h_i(\sigma_i^2) \qquad (2)$$

In several signal processing applications, the above assumptions hold, and further all h_i are *concave* [1, 2]. The objective g is then a concave function of the *subband variance vector* $(\sigma_0^2, \sigma_1^2, \ldots, \sigma_{M-1}^2)$. PCFB's [6, 5, 2], described next, are optimal in all such cases.

1.2 Principal component filter banks (PCFB's)

Definition. Let $\mathcal{C}$ be a class of uniform orthonormal M-channel FB's. A FB in $\mathcal{C}$ is said to be a PCFB for $\mathcal{C}$ for the given input power spectrum if its subband variance vector $(\hat{\sigma}_0^2, \hat{\sigma}_1^2, \ldots, \hat{\sigma}_{M-1}^2)$ **majorizes** the subband variance vector $(\sigma_0^2, \sigma_1^2, \ldots, \sigma_{M-1}^2)$ of every FB in $\mathcal{C}$. This means (by definition of majorization) that after rearranging the σ_i^2 and the $\hat{\sigma}_i^2$ *in decreasing order,*

$$\sum_{i=0}^{P}\hat{\sigma}_i^2 \geq \sum_{i=0}^{P}\sigma_i^2 \quad \text{for} \quad 0 \leq P \leq M-1, \qquad (3)$$

and further equality holds for $P = M - 1$ (this holds automatically for all orthonormal FB's).

Existence of PCFB's. Given the FB class $\mathcal{C}$ and input power spectrum, existence of a PCFB depends on both $\mathcal{C}$ and the input spectrum. For white inputs, every FB

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in $\mathcal{C}$ is a PCFB (for any $\mathcal{C}$). For *two channel* FB's the PCFB is simply the one maximizing the larger subband variance. With $M > 2$ channels, PCFB construction for general input spectra is known only for two special classes $\mathcal{C}$ (*transform coders* and *unconstrained* FB's) [2]. The DFT and cosine–modulated FB classes *do not have* PCFB's for large families of input spectra [2].

PCFB optimality. PCFB's are optimum orthonormal FB's if the minimization objective is a concave function of the subband variance vector. This basic result of [1, 2] leads to PCFB optimality for many FB based signal processing schemes, including progressive transmission, data compression, noise reduction, and as shown recently [8], use in DMT (discrete multitone modulation) communication systems. This motivates generalization of all the above ideas to nonuniform FB's.

2 NONUNIFORM PCFB's: DEFINITIONS AND OPTIMALITY

A nonuniform FB has subband decimation rates that need not be all equal. Figure 2a shows the k-th subband (with decimator n_k) of such a FB. We always assume that (i) n_k are *integers* obeying *maximal decimation* $\sum_k \frac{1}{n_k} = 1$, and (ii) the FB is *orthonormal*, i.e. has PR in absence of subband processing, with $F_k(e^{j\omega}) = H_k^*(e^{j\omega})$.

2.1 Form of minimization objective

Figure 2b shows how *in absence of subband processing*, the nonuniform FB can be redrawn as an equivalent uniform one with decimator $L = \text{lcm}\{n_k\}$. The k-th channel of the nonuniform FB corresponds to $p_k = L/n_k$ channels of the uniform one, in which the filters are delayed versions of each other (see Fig. 2b). The relations between the filters show that (i) a nonuniform FB is orthonormal iff its equivalent uniform FB is so, and (ii) every uniform L band FB *cannot* be redrawn as a nonuniform FB with a given set of decimators n_k (unless we allow its filters to be *periodically time varying* [3]).

Now let processor P_k act on the k-th subband of the nonuniform FB. A general P_k is not representable by independent processors acting on the corresponding p_k subbands of the equivalent uniform FB. However, if all nonuniform subbands are WSS, it happens that for all schemes of [2], P_k is indeed representable in this manner by seperate processors *identical to* P_k (eg. this is clearly true if P_k is a constant multiplier). From Fig. 2b, the inputs to all these p_k processors have same variance σ_k^2 = variance of the k-th subband of the nonuniform FB. Thus from (2) and $L = n_k p_k$, the minimization objective has the form

$$g(\sigma_0^2, \sigma_1^2, \ldots, \sigma_{M-1}^2) = \sum_{i=0}^{M-1} \frac{1}{n_i} h_i(\sigma_i^2) \qquad (4)$$

2.2 Defining nonuniform PCFB's

We aim to define PCFB's for classes of nonuniform FB's with a given set of decimators n_k. The definition for uniform FB's mainly involved the subband variances. For nonuniform FB's, besides the usual subband variance vector $(\sigma_0^2, \sigma_1^2, \ldots, \sigma_{M-1}^2)$, we define the *normalized* and *equivalent uniform* subband variance vectors. The former is defined as $(\frac{\sigma_0^2}{n_0}, \frac{\sigma_1^2}{n_1}, \ldots, \frac{\sigma_{M-1}^2}{n_{M-1}})$; the latter is the subband variance vector of the equivalent uniform FB.

It is desirable that the generalization to nonuniform PCFB's reduce to the usual definition for uniform ones if all n_k are equal; and that the defined PCFB's be optimal for some reasonable class of objectives of the form (4). From this view, the possible natural generalizations are:

1. The *normalized* subband variance vector of the PCFB majorizes that of all FB's in the class.
2. The *equivalent uniform* subband variance vector of the PCFB majorizes that of all FB's in the class.

Majorization here demands in particular that sum of all entries of the variance vector be FB–independent. While this holds automatically due to FB orthonormality, it does not hold if we use the usual subband variance vectors instead of normalized or equivalent uniform ones.

The above superficially similar definitions are in fact distinct: Take the WSS input with spectrum of Fig. 3a, and the class $\mathcal{N}$ having exactly the two FB's FB_I, FB_{II} of Fig. 3b. From the relations among the parameters in Fig. 3a, we can verify that FB_I is the unique PCFB by the first definition, while FB_{II} is the unique PCFB by the second. Thus the class $\mathcal{N}$ here, although artificial, amply illustrates the distinctness of the two definitions.

2.3 Optimality of nonuniform PCFB's

A uniform FB has no intrinsic ordering of its subbands. Permuting the PCFB subbands yields other PCFB's, which may achieve different values for the objective (2). The main result of [2] says that the best of these is optimum when the h_i are concave. Thus an important cause of uniform PCFB optimality besides the majorization property is the fact that we can insert the subband variances in the objective (2) in any order. Each order represents a distinct permutation of the subbands. If we exclude the optimum ordering of the PCFB subbands, the next best one is in general not the next best FB in the given class $\mathcal{C}$. In fact a bad ordering could even yield the worst possible FB for the objective (2).

Now consider nonuniform PCFB's defined using the equivalent uniform subband variance vector. Let α_i be the entries of this vector. The objective (4) is indeed a sum with form $\sum_i f_i(\alpha_i)$ as in (2) (where the f_i can be arranged in groups so that all f_i within any one group are identical to one of the h_i upto scale L). If the α_i could be inserted into this sum in arbitrary order, then indeed the defined PCFB would be optimal whenever all the f_i are concave (i.e. all h_i of (4) are concave).

However the α_i cannot be inserted arbitrarily in the sum $\sum_i f_i(\alpha_i)$: Groups of α_i of equal value, coming from the same subband of the nonuniform FB, must be paired with identical f_i. Thus only certain permutations of the α_i are allowed. The defined PCFB can be claimed to be optimal only if the specific concave h_i are such that the best permutation happens to be one that is allowed. In one notable special case, i.e. when all h_i are identical, the objective is unchanged on permuting the α_i, hence:

Theorem 1. The nonuniform PCFB defined using the equivalent uniform FB is optimal for all objectives of the form (4) when all the h_i are *identical* and *concave.* If the h_i are concave but not identical, the PCFB is optimal only in certain specific situations (explained above).

Identical h_i usually arise from similar processing of all subbands; eg. in reduction of white noise (of variance η^2) when *all* subbands undergo either 0–th order Wiener filtering ($h_i(x) = \frac{x\eta^2}{x+\eta^2}$) or hard thresholding ($h_i(x) = \min(x, \eta^2)$) [2]. Theorem 1 applies in all these situations.

Next we study PCFB's defined using normalized subband variances denoted by β_i. These PCFB's will minimize objectives of form $\sum_i \hat{f}_i(\beta_i)$ for concave $\hat{f}_i$ provided it is allowed to insert the β_i in this sum in any order. Indeed we can rewrite (4) as $\sum_i \hat{f}_i(\beta_i)$ where $\hat{f}_i(x) = (1/n_i)h_i(n_i x)$. However, reordering the β_i now requires corresponding changes in the $\hat{f}_i$ too, as $\hat{f}_i$ depends on the decimator n_i. So again the PCFB will not be optimal in general even if all h_i are concave. In one special case, i.e. $h_i(x) = k_i x$ for constant k_i, we have $\hat{f}_i \equiv h_i$ which is concave *and* independent of n_i; hence:

Theorem 2. The nonuniform PCFB defined using normalized variances is optimal for all objectives of the form (4) when for all i, $h_i(x) = k_i x$ for constant k_i.

Theorem 2 applies in data compression [2], when the i-th subband quantizer is modelled as additive noise with variance $k_i\sigma_i^2$. Here the constant k_i depends on the number of bits alloted to the i-th quantizer.

3 EXISTENCE OF NONUNIFORM PCFB's

General existence results for nonuniform PCFB's defined using normalized variances are currently unknown. In this section, nonuniform PCFB's always mean those defined using equivalent uniform FB's. The FB class $\mathcal{N}$ studied here consists of *all* (unconstrained) orthonormal nonuniform FB's having a given set of decimators $n_0, n_1, \ldots, n_{M-1}$ with lcm L. Using our knowledge of PCFB's for the class $\mathcal{C}$ of unconstrained *uniform* L band FB's, we show some existence results for PCFB's for $\mathcal{N}$.

Let $\mathcal{C}_{\mathcal{N}} \subset \mathcal{C}$ be the class of uniform FB's equivalent to FB's in $\mathcal{N}$. By our definition, PCFB's for $\mathcal{C}_{\mathcal{N}}$ and $\mathcal{N}$ are equivalent. Now if a FB in $\mathcal{C}_{\mathcal{N}}$ has subbands obeying total decorrelation and spectral majorization [7], it is a PCFB for $\mathcal{C}$, and hence for $\mathcal{C}_{\mathcal{N}} \subset \mathcal{C}$. These facts lead to:

Theorem 3. Any FB with totally decorrelated and *white* subbands is a PCFB for $\mathcal{N}$. For example, a FB with nonoverlapping brickwall analysis filters is a PCFB if the input spectrum is constant on the support of each filter (Fig. 4a).

The conditions of Theorem 3 ensure that the equivalent uniform FB is a PCFB for $\mathcal{C}$. This is because if a subband of the nonuniform FB is white, from Fig. 2 we can show that the corresponding subbands of the equivalent uniform FB are also white and totally decorrelated. More generally, if the nonuniform subband spectrum has form $A(e^{j\omega p_k})$, the corresponding uniform FB subbands are totally decorrelated with identical spectra $A(e^{j\omega})$. Using this we can create more complex input spectra for which brickwall FB's are PCFB's (Fig. 4b). We conclude with a nonexistence result proved in [4]:

Theorem 4. Let the input spectrum $S(e^{j\omega})$ be *strictly monotonic* (increasing or decreasing) for $a \leq \omega < a + 2\pi$ for some real a. Then there is no PCFB for the unconstrained nonuniform FB class $\mathcal{N}$ with any given set of decimators *not all equal.*

PCFB's exist for many spectra $S(e^{j\omega})$ that are monotonic for $a \leq \omega < a + 2\pi$ for some a, eg. the flat (white) spectrum or those with form as in Fig. 4a. By slight perturbations of such spectra, we can cause them to become strictly monotone for $a \leq \omega < a + 2\pi$, hence (by Theorem 4) destroying the existence of a PCFB. Thus, existence of nonuniform PCFB's can be very delicate.

References

[1] S.Akkarakaran and P.P.Vaidyanathan, "On optimization of filter banks with denoising applications," in *Proc. IEEE ISCAS,* Orlando, FL, 1999.

[2] S.Akkarakaran and P.P.Vaidyanathan, "Filter bank optimization with convex objectives, and the optimality of principal component forms," *submitted to IEEE Trans. Signal Processing.*

[3] S.Akkarakaran and P.P.Vaidyanathan, "New results and open problems on nonuniform filter banks," in *Proc. IEEE ICASSP,* Phoenix, AZ, Mar. 1999.

[4] S.Akkarakaran and P.P.Vaidyanathan, "On nonuniform principal component filter banks: Definitions, existence and optimality," in *Proc. SPIE,* San Diego, CA, 2000.

[5] M.K.Tsatsanis and G.B.Giannakis, "Principal component filter banks for optimal multiresolution analysis," *IEEE Trans. SP,* pp. 1766-1777, Aug. 1995.

[6] M.Unser, "An extension of the KLT for wavelets and perfect reconstruction filter banks," in *Proc. SPIE no. 2034,* San Diego, CA, 1993, pp.45-56.

[7] P.P.Vaidyanathan, "Theory of optimal orthonormal subband coders," *IEEE Trans. SP,* vol. 46, no. 6, pp. 1528-1543, June 1998.

[8] P.P.Vaidyanathan, Y-P.Lin, S.Akkarakaran and S-M.Phoong, "Optimizing the capacity of orthogonal and biorthogonal DMT channels," in *Proc. EUSIPCO,* Tampere, Finland, Sept. 2000.

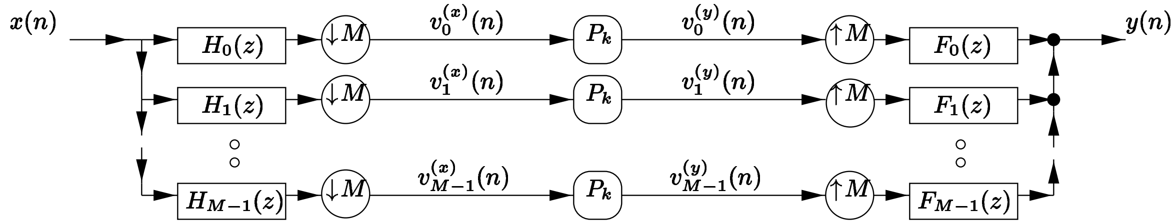


Fig. 1. General subband signal processing scheme using a uniform M channel filter bank.

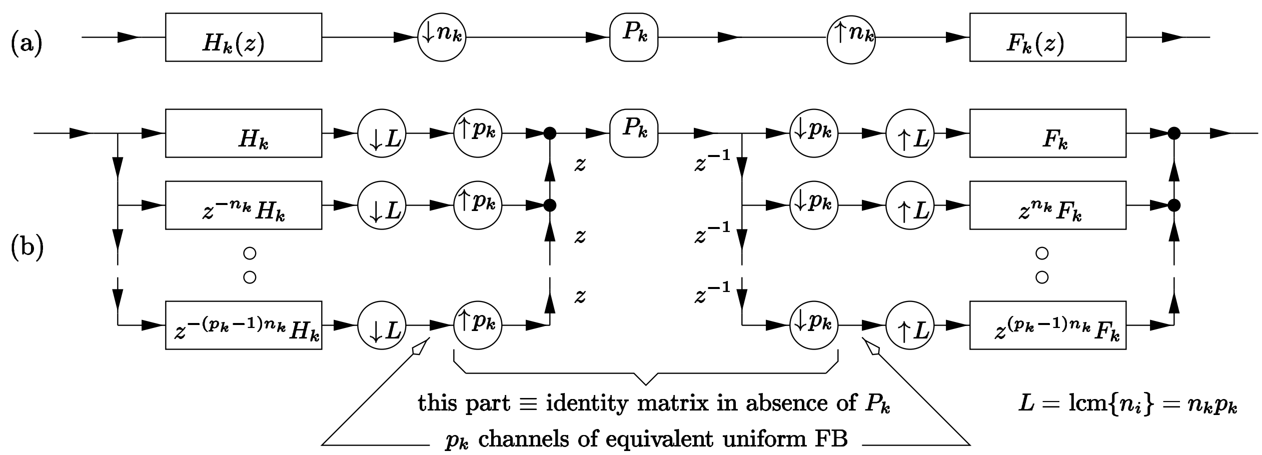
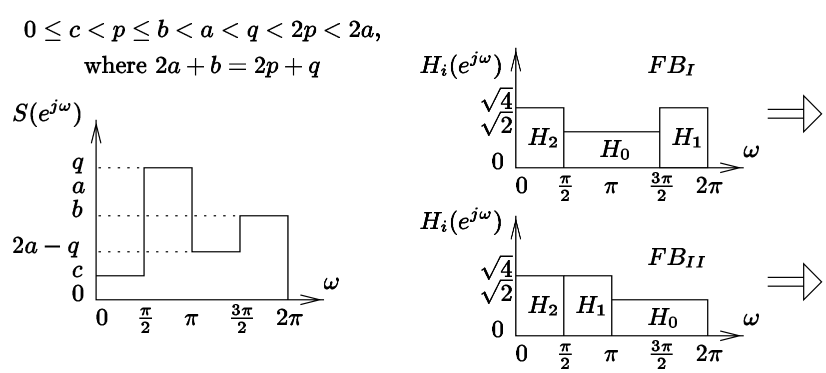


Fig. 2. Nonuniform FB's. (a) A single subband, (b) corresponding subbands of equivalent uniform FB.

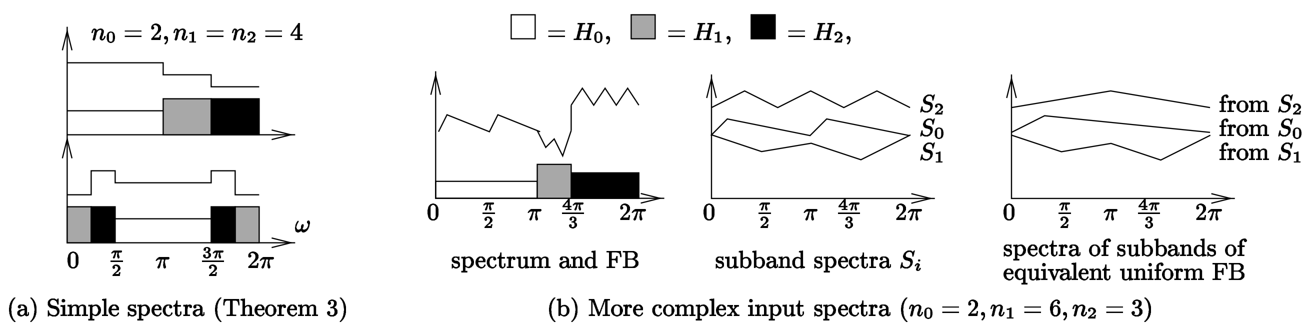


Subband variance vectors

	usual	normalized	equivalent uniform
FB_I	(a,b,c) $a > b > c$	$(\frac{a}{2},\frac{b}{4},\frac{c}{4})$ $\frac{a}{2} > \frac{b}{4} > \frac{c}{4}$	(a,a,b,c) $a = a > b > c$
FB_{II}	(p,q,c) $q > p > c$	$(\frac{p}{2},\frac{q}{4},\frac{c}{4})$ $\frac{p}{2} > \frac{q}{4} > \frac{c}{4}$	(p,p,q,c) $q > p = p > c$

(a) Input spectrum $S(e^{j\omega})$ (b) FB's with decimators $n_0 = 2, n_1 = n_2 = 4$

Fig. 3. Distinctness of two definitions of nonuniform PCFB's.



(a) Simple spectra (Theorem 3) (b) More complex input spectra ($n_0 = 2, n_1 = 6, n_2 = 3$)

Fig. 4. Nonuniform PCFB's for certain input spectra.