# ON OPTIMAL TRANSFORMS FOR SUBBAND DOMAIN SUPPRESSION OF COLORED NOISE

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## ABSTRACT

Suppose we wish to analyze a noisy signal using a filter bank (FB) and apply noise suppression schemes such as Wiener filters in the subbands of the FB. This paper formalizes and studies the problem of finding the best FB for this purpose. The best FB depends on the class of allowed FB's, the type of subband processing, and the statistics of the input signal and additive noise. Recently we have shown the optimality of the so–called *principal component* filter bank (PCFB) for several signal processing problems. In particular, the PCFB is the optimum orthonormal FB for many schemes for suppression of *white* noise. With colored noise however, the optimization is considerably more involved, and PCFB optimality is much more restricted. Here we present several results on the colored noise suppression problem. We develop an algorithm to find the exact globally optimum unconstrained orthonormal FB for piecewise constant input signal and noise spectra. This thus allows approximation of the optimum FB for any spectra to any desired accuracy. We examine the role of PCFB's in the optimization.

## 1. PROBLEM FORMULATION

Figure 1 shows an $M$ band uniform filter bank (FB) used for noise reduction. The FB input $x(n)$ is the sum of a pure signal $s(n)$ and noise $\mu(n)$, both wide sense stationary (WSS) random processes with given power spectral densities (psd's). The noise $\mu(n)$ is zero mean and uncorrelated to $s(n)$. We only study uniform *orthonormal* FB's, i.e., those as in Fig. 1 where $H_i(e^{j\omega}) = F_i^*(e^{j\omega})$ and the output is identical to the input *in absence of subband processing.*

The subband noise suppression systems we consider are memoryless multipliers $k_i$ which may or may not depend on their input statistics. The error signal $e(n) = y(n) - s(n)$ between the true and desired FB output is then wide sense cyclostationary with period $M$. Equivalently, the subband errors $v_i^{(e)}(n) = k_i v_i^{(x)}(n) - v_i^{(s)}(n)$ are jointly WSS. Due to FB orthonormality, the average of the variances of the $v_i^{(e)}(n)$ is the mean square value of the error $e(n)$ averaged over its period of cyclostationarity $M$. We wish to **choose a FB minimizing this mean square error** over all possible FB's in the given class $\mathcal{C}$ of uniform orthonormal $M$ band FB's, for the given input signal and noise psd's.

Let $\sigma_i^2, \eta_i^2$ respectively be the variances of $v_i^{(s)}(n), v_i^{(\mu)}(n)$. Thus $\mathbf{v}_\sigma = (\sigma_0^2, \sigma_1^2, \ldots, \sigma_{M-1}^2)^T$, $\mathbf{v}_\eta = (\eta_0^2, \eta_1^2, \ldots, \eta_{M-1}^2)^T$ are respectively the signal and noise subband variance vectors. In many cases, computing the subband error variances shows that the minimization objective has the form

$$f(\mathbf{v}_\sigma, \mathbf{v}_\eta) = \frac{1}{M}\sum_{i=0}^{M-1} f_i(\sigma_i^2, \eta_i^2), \qquad (1)$$

where $f_i$ depends on the specific choice of $k_i$. Examples are:

| $k_i$ choice | process type | $f_i(x,y)$ |
|---|---|---|
| $\sigma_i^2/(\sigma_i^2+\eta_i^2)$ | $0^{th}$ order Wiener | $xy/(x+y)$ |
| $\begin{cases} 1 & \text{if } \sigma_i^2 \geq \eta_i^2 \\ 0 & \text{otherwise} \end{cases}$ | hard threshold | $\min(x,y)$ |
| constant | FB independent | $x\|1-k_i\|^2 + y\|k_i\|^2$ |

All these $f_i$ are verifiable to be *concave* on $\mathcal{R}_+^2$ (the non-negative quadrant of $\mathcal{R}^2$, i.e., $\{(x,y) : x, y \geq 0\}$). Thus the objective (1), as a function of the vector $\mathbf{v} = (\mathbf{v}_\sigma^T, \mathbf{v}_\eta^T)^T$, is *concave* on $\mathcal{R}_+^{2M}$. This is crucial to our methods of finding the best FB, i.e., the choice of $\mathbf{v}$ minimizing (1).

## 2. EARLIER WORK, OVERVIEW OF RESULTS

Suppose the noise is white, i.e., has constant psd. Then all entries $\eta_i^2$ of $\mathbf{v}_\eta$ equal the input noise variance, independent of the FB; so the objective depends only on $\mathbf{v}_\sigma$. Let the signal spectrum be constant on each interval $\mathcal{I}_k = [\frac{2\pi k}{M}, \frac{2\pi(k+1)}{M})$ for integer $k$. Then the standard contiguous stacked *brickwall* $M$ band FB (whose $k$-th filter $H_k(e^{j\omega})$ is $\sqrt{M}$ on $\mathcal{I}_k$ and zero on $[0, 2\pi)\backslash\mathcal{I}_k$) yields white subband signal component processes totally uncorrelated to each other. It is intuitively very appealing to believe that this FB is the best possible one, i.e., is optimum in the class $\mathcal{C}^u$ of unconstrained orthonormal $M$ band FB's (FB's whose filters have no constraints besides those imposed by orthonormality). This is indeed true: This FB is a *principal component* filter bank (PCFB) [1, 2, 8, 9, 10] for the assumed signal psd, and its optimality follows from our recent work [1, 2]. This can be summarized as follows:

**Fact 1.** [7, 5] Let us be given a set $D$ whose convex hull (denoted by $\text{co}(D)$) is compact, and a function $g$ that is

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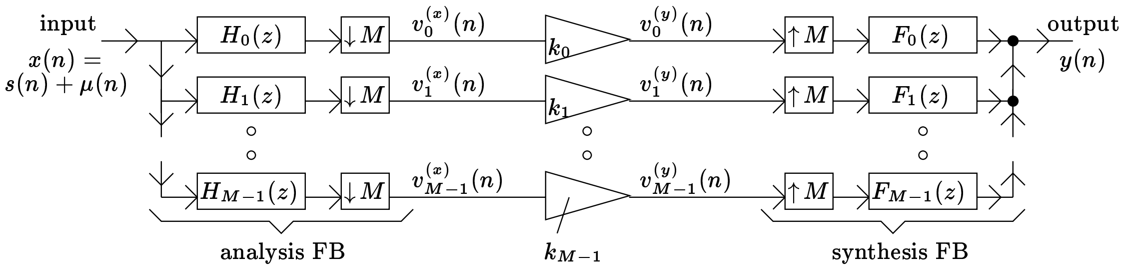


**Fig. 1: Filter bank used for noise suppression.**

concave on $\mathrm{co}(D)$. The minimum of $g$ over $D$ can be found by minimizing $g$ over the set $E$ of extreme points of $\mathrm{co}(D)$. (We always have $E \subset D$.) In particular if $\mathrm{co}(D)$ is a *polytope*, i.e., a (compact) convex set with *finitely many* extreme points, then this is simply a finite search over these points.

**Fact 2.** [1, 2] Given a class $\mathcal{C}$ of uniform orthonormal $M$ band FB's and the psd of the WSS input, let $S$ be the set of subband variance vectors obtained by passing the input through each FB in $\mathcal{C}$. A PCFB for $\mathcal{C}$ for the given input psd exists iff $\mathrm{co}(S)$ is a polytope whose set of extreme points is the set of all permutations of a single vector $\mathbf{v}_*$. Under this condition, $\mathbf{v}_*$ corresponds to the PCFB (its permutations represent the permutations of the subbands of the PCFB).

Facts 1 and 2 imply that the PCFB minimizes all concave functions of the subband variance vector. PCFB construction for given input psd is well studied for the unconstrained class $\mathcal{C}^u$ [10], the *transform coder* class and any two–channel FB class [2]. Thus, the problem of Section 1 has been solved for these classes when the noise is white.

Now let the noise be colored, and let both the signal and noise psd's be constant on each interval $\mathcal{I}_k = [\frac{2\pi k}{MN}, \frac{2\pi(k+1)}{MN})$ for integers $k$, for some fixed integer $N$. If $N = 1$, again it is intuitively appealing to believe that the usual contiguous stacked brickwall FB is optimal. We prove this fact, and further show how to find the best FB *for any N*.

When $N = 1$, the usual brickwall FB is a *common* PCFB for both the signal and noise. Even for FB classes such as $\mathcal{C}^u$, where individual PCFB's for the signal and noise always exist, existence of common PCFB's is not always ensured: It depends on the psd's. However, the result suggests that perhaps such a common PCFB is always optimal *if* it exists. This happens to be true *for a particular FB class*, namely the class of transform coders [3]. However, for general FB classes it is true only for certain restricted families of input psd's – examples being (a) the earlier mentioned case of piecewise constant psd's, for the FB class $\mathcal{C}^u$; and (b) the case when the noise is white, for any FB class (all FB's are PCFB's for a white input, no matter what the FB class). The potential suboptimality of the common PCFB is another corollary of the results of this paper.

## 3. THE MAIN RESULT

### 3.1. Statement of the result

**Theorem 1.** Consider optimization of FB's in the class $\mathcal{C}^u$ of unconstrained uniform orthonormal $M$ band FB's, for the noise suppression systems of Section 1. The minimization objective $f$ is given to be a concave function of the vector $\mathbf{v} = (\mathbf{v}_\sigma^T, \mathbf{v}_\eta^T)^T$ of signal and noise subband variances. Let $S_v$ denote the 'search space' defined as the set of all possible values of $\mathbf{v}$ (corresponding to all FB's in $\mathcal{C}^u$, for the given input psd's). Let $E_v$ be the set of extreme points of $\mathrm{co}(S_v)$ (the convex hull of $S_v$). From Fact 1, it suffices to minimize $f$ over the set $E_v$. Suppose the input signal and noise psd's are constant on all intervals $(\frac{2\pi k}{MN}, \frac{2\pi(k+1)}{MN})$ for all integers $k$ for some fixed positive integer $N$. Then,

1. $S_v$ is a polytope, i.e., $S_v = \mathrm{co}(S_v)$ and $E_v$ is finite. Further, let $\mathcal{F}$ be the set of all *brickwall* FB's in $\mathcal{C}^u$ having all filter band edges at integer multiples of $\frac{2\pi}{MN}$. Then $\mathcal{F}$ has $|\mathcal{F}| = (M!)^N$ FB's, and for each point of $E_v$ there is a FB in $\mathcal{F}$ corresponding to it.
2. The size of $\mathcal{F}$ is exponential in $N$, but the number of FB's in $\mathcal{F}$ actually corresponding to points in $E_v$ is only polynomial in $N$, i.e., $|E_v| \le K_M N^{2M-3}$. However, both $|\mathcal{F}|$ and $|E_v|$ are super–exponential in $M$, as $K_M = 4(2M-3)(M!(M!-1)/2)^{2M-3}$.
3. The number of arithmetic operations needed to extract from $\mathcal{F}$ the FB's corresponding to points in $E_v$ is also polynomial in $N$, though it is super–exponential in $M$. It does not exceed $C_M N^{2M-2}$ if $M > 2$ and $DN \log N$ if $M = 2$, where $C_M = G(M!)^{4M-5}$ and $G, D$ are constants. ∎

The result is proved in full in [3]. Here in Section 3.3, we present the full proof except for the justification of a technical lemma it needs. Section 3.4 elaborates on the algorithm that extracts from $\mathcal{F}$ the FB's corresponding to extreme points of $S_v$ (implied by item (3) of the theorem).

### 3.2. Discussion on Theorem 1

*Result appealing but not obvious:* FB's in $\mathcal{F}$ are brickwall with nonoverlapping filter responses having shapes similar to the input psd's (piecewise constant with the same allowed discontinuities). That the best FB can always be chosen to be in $\mathcal{F}$ is hence very appealing. However it is certainly not obvious; e.g. it is not true if the objective is not concave.

*Bounds on $|E_v|$:* Items (1) and (2) of Theorem 1 set on the size of $E_v$, the bounds $(M!)^N (= |\mathcal{F}|)$ and $K_M N^{2M-3}$, respectively the stronger for the cases $M \gg N$ and $N \gg M$.

*Common PCFB's and the $N = 1$ case:* Theorem 1 holds whether or not a common signal and noise PCFB for $\mathcal{C}^u$ exists for the given psd's. If it exists, it also corresponds to points of $E_v$ (often it too is in $\mathcal{F}$). However it need not always be optimal, as $E_v$ could in general have other points too (see [3] for a specific instance of suboptimality). If $N = 1$ however, all $M!$ elements of $\mathcal{F}$ are permutations of the same FB, namely the usual contiguous–stacked brickwall FB, which is hence always optimal. This result was noted in Section 2.

*Approximating optimum FB's for arbitrary spectra:* Most spectra can be approximated by the piecewise constant ones in the premise of Theorem 1, to any desired accuracy by sufficiently increasing $M$ and/or $N$. Thus Theorem 1 in principle allows approximation of the optimum FB in $\mathcal{C}^u$ for any input spectra to any desired accuracy. However the complexity of the algorithm for this is polynomial in $N$ but super–exponential in $M$. Thus, we have good algorithms for low $M$ (especially $M = 2$, where the complexity is of order $N \log N$). For sufficiently large $M$, we get good enough approximations of the true spectra by taking $N = 1$. The earlier remark then gives at no cost, the optimum FB in $\mathcal{C}^u$ , i.e., the usual contiguous–stacked brickwall FB.

### 3.3. Proof of Theorem 1

Let $H_i(e^{j\omega}), i = 0, 1, \ldots, M-1$ be the analysis filters of a $M$-channel orthonormal FB (i.e., a FB from $\mathcal{C}^u$ ). For $l = 0, 1, \ldots, N-1$, define $M \times M$ matrices $\mathbf{G}^l$ whose $ik$-th entries ($i, k = 0, 1, \ldots, M-1$) are

$$g^l_{ik} \triangleq \frac{N}{2\pi} \int_{\frac{2\pi(l+Nk)}{MN}}^{\frac{2\pi(l+Nk+1)}{MN}} \left|H_i(e^{j\omega})\right|^2 \, d\omega. \tag{2}$$

Let the constant values of the input signal and noise psd's $D_s(e^{j\omega})$ and $D_\mu(e^{j\omega})$ on the interval $(\frac{2\pi k}{MN}, \frac{2\pi(k+1)}{MN})$ be $a_k, b_k$ respectively. Let $\sigma_i^2, \eta_i^2$ respectively be the signal and noise variances in the $i$-th subband. Then

$$\sigma_i^2 = \frac{1}{2\pi} \int_0^{2\pi} \left|H_i(e^{j\omega})\right|^2 D_s(e^{j\omega}) \, d\omega = \frac{1}{N} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g^l_{ik} a_{l+Nk}.$$

Likewise, $N\eta_i^2 = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g^l_{ik} b_{l+Nk}$. Let $\mathcal{G}$ be the set of ordered sets $\underline{\mathbf{G}} = (\mathbf{G}^0, \mathbf{G}^1, \ldots, \mathbf{G}^{N-1})$ corresponding to all FB's in $\mathcal{C}^u$ . From the computed $\sigma_i^2, \eta_i^2$, the variance vector $\mathbf{v} \in S_v$ corresponding to $\underline{\mathbf{G}} \in \mathcal{G}$ is then

$$\mathbf{v} = \mathcal{L}(\underline{\mathbf{G}}) = \sum_{l=0}^{N-1} \begin{pmatrix} \mathbf{G}^l \mathbf{a}^l \\ \mathbf{G}^l \mathbf{b}^l \end{pmatrix}, \tag{3}$$

where $\mathbf{a}^l = (a_l, a_{l+N}, \ldots, a_{l+(M-1)N})^T$, and $\mathbf{b}^l$ is similarly defined using the $b_k$. Thus $S_v$ is the image of $\mathcal{G}$ under a linear map $\mathcal{L}$. Hence we study $\mathcal{G}$. Integrating over appropriate intervals the relations $\sum_{i=0}^{M-1} \left|H_i(e^{j\omega})\right|^2 = M$ and $\sum_{k=0}^{M-1} \left|H_i(e^{j(\omega+\frac{2\pi k}{M})})\right|^2 = M$ (which hold for all $\omega$ for any orthonormal $M$ band FB [10]) gives

$$0 \le g^l_{ik} \le 1, \quad \sum_{i=0}^{M-1} g^l_{ik} = \sum_{k=0}^{M-1} g^l_{ik} = 1 \tag{4}$$

for all $i, k, l$ for which $g^l_{ik}$ are defined. This is by definition the statement that $\mathbf{G}^l$ is **doubly stochastic** for all $l =$ $0, 1, \ldots, N-1$. Let $\mathcal{Q}, \mathcal{P}$ respectively be the sets of all $M \times M$ doubly stochastic matrices and permutation matrices. We have thus shown that $\mathcal{G} \subset \mathcal{Q}^N = \mathcal{Q} \times \mathcal{Q} \times \ldots \times \mathcal{Q}$.

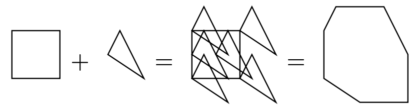

**Fig. 2: Minkowski sum in two dimensions**

*Claim:* $\mathcal{G} = \mathcal{Q}^N$, which (by Birkhoff's theorem [2, 5]) is a polytope with $\mathcal{P}^N$ as its set of extreme points. Also, FB's in the set $\mathcal{F}$ (defined in stating Theorem 1) correspond directly (one–to–one) with points in $\mathcal{P}^N$.

We refer to [3] for a proof of this claim, which involves building a brickwall FB in $\mathcal{C}^u$ corresponding to any given $\underline{\mathbf{G}} \in \mathcal{Q}^N$. Once the claim is established, we have then proved item (1) in the statement of Theorem 1: Recall that $S_v$ is the image of $\mathcal{G}$ under a linear map $\mathcal{L}$ of (3). The linear image of a polytope $D$ is a polytope $R$ whose extreme points are all images of some extreme points of $D$. The claim above thus means that there is a FB in $\mathcal{F}$ for every extreme point of $S_v$. From the correspondence between $\mathcal{F}$ and $\mathcal{P}^N$, $\mathcal{F}$ has $|\mathcal{F}| = \left|\mathcal{P}^N\right| = |\mathcal{P}|^N = (M!)^N$ FB's (counting separately all permutations of each FB in $\mathcal{F}$ – else we must divide the number by $M!$).

*Proof of items 2,3 of Theorem 1 statement:* For any fixed $l = 0, 1, \ldots, N-1$, as $\mathbf{G}^l$ can be any element of $\mathcal{Q}$, the set of possible values of $\mathbf{x}^l = \begin{pmatrix} \mathbf{G}^l \mathbf{a}^l \\ \mathbf{G}^l \mathbf{b}^l \end{pmatrix}$ is itself a polytope $T^l$ (as it is a linear image of $\mathcal{Q}$). It lies on a $2M-2$ dimensional hyperplane in $\mathcal{R}^{2M}$, as the first and last $M$ entries of $\mathbf{x}^l$ have sums independent of $\mathbf{G}^l \in \mathcal{Q}$ (respectively the sums of entries of $\mathbf{a}^l, \mathbf{b}^l$). The extreme points of $T^l$ correspond to the $M!$ possible choices of $\mathbf{G}^l \in \mathcal{P}$. From (3), $S_v = \left\{\sum_l \mathbf{x}^l \;:\; \mathbf{x}^l \in T^l\right\}$, which is known as the *Minkowski sum* [4],[6] of the polytopes $T^l$. Theorem 2.1.10 and Corollary 2.1.11 of [4] bound the number of extreme points of the Minkowski sum of $k$ polytopes of dimension $d$ with not more than $p$ extreme points each. Theorem 2.3.7$'$ and Proposition 2.3.9 of [4], with their proofs, outline algorithms to find the extreme points of this Minkowski sum, thus bounding the number of arithmetic operations needed for the same. Applying these bounds with $k = N$, $d = 2M-2$ and $p = M!$ yields items 2,3 of the statement of Theorem 1. $\triangledown\triangledown\triangledown$

### 3.4. The Minkowski sum algorithm

As seen above, the Minkowski sum of a finite number of sets $T_i \subset \mathcal{R}^d$ is the set $\{\sum_i \mathbf{x}_i : \mathbf{x}_i \in T_i\}$. Such sums have been well studied in computational geometry, e.g. in context of robot motion planning algorithms [6] when $d = 2, 3$. Figure 2 illustrates the Minkowski sum of two polygons in $\mathcal{R}^2$. This section describes the principles of the algorithms of [4] used above to compute Minkowski sums of polytopes, and illustrates their role in Theorem 1 using the case of $M = 2$.

It is not hard to see that the Minkowski sum $S$ of $k$ polytopes is a polytope, in fact, specifically it is the convex

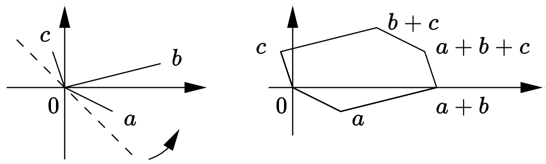


**Fig. 3: Showing Minkowski sum algorithm.**

hull of the (finite) Minkowski sum $V$ of the sets of extreme points of the individual polytopes. (See Fig. 2 for an illustration.) However, if the summand polytopes of $S$ have $<= p$ extreme points each, then in general $V$ has $p^k$ points. Not all of these are extreme, the true number of extreme points is only polynomial in $p, k$. The key idea in finding these extreme points is the following:

**Fact 3.** [7] A point $\mathbf{v} = (v_0, v_1, \ldots, v_{d-1})$ in a polytope $P \subset \mathcal{R}^d$ is an extreme point of $P$ iff there is a linear function $f$ on $\mathcal{R}^d$ (i.e., $f$ of form $f(\mathbf{v}) = \sum_i a_i v_i$ for real constants $a_i$) which is maximized over $P$ uniquely by $\mathbf{v}$.

Any such linear $f$ is always maximized over $P$ by at least one extreme point of $P$, and by a slight perturbation of $f$ (i.e., of the $a_i$) we can ensure uniqueness of the maximum. Further if $S$ is a Minkowski sum, then its definition and the linearity of $f$ imply that $\mathbf{v}$ is a maximum of $f$ over $S$ iff it is obtainable by choosing from each summand polytope a vector maximizing $f$ over that summand, and adding the chosen vectors. So we choose all possible $f$ that have a unique maximum over each summand polytope, and for each $f$, add these maxima to obtain an extreme point of $S$. Many $f$ yield the same extreme point (there are infinitely many $f$ but finitely many extreme points), so we need a scheme to select a set of $f$ that will yield all the extreme points. A formal description and analysis of complexity of such a scheme can be found in [4], using the concept of a *normal fan* of a polytope. Here we merely illustrate the process as applied in Theorem 1 for $M = 2$.

When $M = 2$, Section 3.3 shows that the search space $S_v$ is the Minkowski sum of $N$ two–dimensional polytopes, each with two vertices; i.e., of $N$ line segments in $\mathcal{R}^2$. Adding a constant vector to each element in a summand polytope has the same effect on the Minkowski sum. So without losing generality we can do this for each segment and assume the origin to be one endpoint. Let the other endpoints of the segments be $\mathbf{v}_i, i = 0, 1, \ldots, N-1$. The linear functions on $\mathcal{R}^2$ are (upto scale factor) the projections along the various lines through the origin. Thus, for each such line, we sum all the $\mathbf{v}_i$ with a positive projection (i.e., lying on one of the two half–planes associated with the line) to get an extreme point of the Minkowski sum. We then rotate the selected line about the origin until it crosses over a point $\mathbf{v}_i$, upon which the computed extreme point changes, i.e., is incremented by $\pm\mathbf{v}_i$ depending on whether $\mathbf{v}_i$ moved from the left to the right half plane or vice-verca. We do this operation until the line completes a full 360 degree rotation. Figure 3 shows the process for $N = 3$ line segments. As the dotted line rotates anticlockwise from its position in the figure, we obtain all 6 vertices of the Minkowski sum in anticlockwise order starting from the vertex $a + b + c$.

If the vertex vectors $\mathbf{v}_i$ are sorted in increasing order of angle with respect to a given fixed direction, then the number of arithmetic operations required by the above process is linear in the size of this sorted list. Also the process gives the exact number of extreme points in the Minkowski sum, namely twice the total number of nonparallel line segments being added, in general equal to $2N$ (improving the bound $4N$ of Theorem 1). The order of complexity of the algorithm is reported as linear in $N$ in [4], and more precisely as $N \log N$ in Theorem 1, which includes sorting of the $\mathbf{v}_i$. The general complexity bound for all $M$ (Theorem 1, item 3) implies an order of $N^2$ for $M = 2$, the reason why this could be improved is because in $\mathcal{R}^2$ there is a canonical way to sweep in sequence through all the linear functions $f$.

## 4. CONCLUSION

We have developed an algorithm to find the optimum orthonormal $M$ band FB for noise suppression when both the signal and noise spectra are piecewise constant with discontinuities at rational multiples of $\frac{2\pi}{MN}$. The algorithm can be used to approximate the optimum solution for arbitrary spectra if $M$ and/or $N$ is large. For large $M$, it suffices to assume $N = 1$, in which case the usual contiguous–stacked brickwall FB is optimal.

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