NEW INSIGHTS INTO MULTIRATE SYSTEMS WITH STOCHASTIC INPUTS USING BIFREQUENCY ANALYSIS

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ABSTRACT

In multirate processing, it is often necessary to understand how the statistical properties of signals (such as stationarity) are altered by passage through multirate systems. Some of these issues have been addressed in [1]. For example, it is shown in [1] that a necessary and sufficient condition for the output of an L-fold interpolation filter to be wide sense stationary (WSS) for all WSS inputs, is that the filter have an aliasfree(L) support. However this result was established using conventional tools such as polyphase matrices, which resulted in a convoluted derivation which does not provide much insight. It also leaves many questions unanswered, since it does not generalize easily to the case of systems with vector inputs. This paper shows that problems of this nature can be addressed in an elegant and insightful manner by using **bifrequency maps** [2],[3], and **bispectra** [4]. In particular, we give a simpler proof of the above-mentioned result of [1], and generalize it to the case of vector systems.

1. INTRODUCTION

Consider the L-fold scalar interpolation filter shown in Fig. 1. For this system, [1] derives a necessary and sufficient condition for the output to be wide sense stationary (WSS) for all WSS inputs. The condition is that the filter $H(e^{j\omega})$ have aliasfree(L) support. The derivation of this result was based on the fact that a scalar random process is WSS iff its blocked version is WSS with pseudocirculant power spectral density (psd) matrix. The use of pseudocirculants however results in very implicit conditions on the system. Transforming them into the explicit aliasfree(L) condition is laborious, and does not provide much insight. Further, it becomes extremely difficult to generalize this approach to the case of multi-input multi-output (MIMO) systems with vector random process inputs.

This paper shows that problems of this nature can be solved very elegently and in a geometrically insightful way using bifrequency maps [2],[3] and bispectra [4]. These are two-dimensional Fourier transforms that characterize all linear time-varying (LTV) systems and nonstationary random processes respectively. Though these tools have been known in the literature, they have not often been used for analysis of multirate systems, being very general. However they greatly simplify the analysis in the context of problems of the kind mentioned above. We provide a much simplified derivation of the above-mentioned aliasfree(L) condition, and extend it to the MIMO case shown in Fig. 2.

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2. BIFREQUENCIES AND BISPECTRA

2.1. Bifrequencies and LPTV systems

A MIMO LTV system [2] with input $\mathbf{x}(n)$ and output $\mathbf{y}(n)$ is fully specified by the time-domain input-output relation

$$\mathbf{y}(m)=\sum_{n=-\infty}^{\infty}\mathbf{k}(m,n)\mathbf{x}(n)=\sum_{n=-\infty}^{\infty}\mathbf{h}(m,n)\mathbf{x}(m-n) \quad (1)$$

Here $\mathbf{k}(m,n)$ is called the Green's function, and is perfectly general; while $\mathbf{h}(m,n)$ is the time-varying impulse response, useful only if the input and output rates are equal [2]. These are related as $\mathbf{h}(m,n)=\mathbf{k}(m,m-n)$. The bifrequency function of the LTV system is the 2D Fourier transform of the Green's function, i.e.

$$\mathbf{K}(e^{j\omega'},e^{j\omega})=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\mathbf{k}(m,n)e^{-j\omega' m}e^{j\omega n} \quad (2)$$

The LTV system is said to be LPTV(L) (linear periodically time-varying with period L) if $\mathbf{h}(m,n)$ is periodic in m with period L for each n. Such a system can be shown [5, chapter 10] to be equivalent to a uniform L-channel maximally decimated filter-bank. For such a system, the bifrequency function can be shown [6] to consist of a set of parallel impulsive lines, as illustrated by Fig. 3a. The q-th line is the line $\omega'-\omega=2\pi q/L$. The shape of the impulse along this line is given by a transfer function denoted by $F_q(e^{j\omega})$, which has a periodicity of L in q. For $q=0,1,\ldots,L-1$ we have $F_q(e^{j\omega})=A_q(e^{j\omega})$ where $A_q(e^{j\omega})$ are the aliasing gains [5, chapter 5] of the filter-bank, that describe the input output relation according to

$$Y(e^{j\omega})=\sum_{q=0}^{L-1}A_q(e^{j\omega})X(e^{j(\omega-\frac{2\pi q}{L})}) \quad (3)$$

Bifrequency maps have been used in [2] to elegantly explain the action of decimators and expanders on deterministic inputs. This paper shows that the same can be done with stochastic inputs (not considered in [2]), which leads to results that would be otherwise hard to see.

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2.2. Bispectra and CWSS processes

The autocorrelation matrix $\mathbf{r_x}(m,n)$ and bispectrum matrix $\mathbf{S_x}(e^{j\omega'},e^{j\omega})$ of a nonstationary vector process $\mathbf{x}(n)$ are defined as[1]

$$\mathbf{r_x}(m,n) = E[\mathbf{x}(m)\mathbf{x}^\dagger(n)] \tag{4}$$

$$\mathbf{S_x}(e^{j\omega'},e^{j\omega}) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\mathbf{r_x}(m,n)e^{-j\omega' m}e^{j\omega n} \tag{5}$$

Bispectra have been studied in [4], [9] where some of their properties have been stated for the case of continuous-time scalar processes. These hold for the bispectrum $S_x(e^{j\omega'},e^{j\omega})$ of discrete-time scalar random processes too. For example, while $S_x(e^{j\omega'},e^{j\omega})$ may not be real in general, $S_x(e^{j\omega},e^{j\omega})$ is real and non-negative for all real ω. More generally for vector processes, $\mathbf{S_x}(e^{j\omega},e^{j\omega})$ is Hermitian positive semidefinite for all real ω. Analogous to the case of LPTV(L) systems, a CWSS(L) (cyclo-wide-sense stationary with period L) process $\mathbf{x}(n)$ is one for which $E[\mathbf{x}(m)\mathbf{x}^\dagger(m-n)]$ is periodic in m with period L for each n. Its bispectrum matrix consists of a set of parallel impulsive lines as in Fig. 3a. The explicit expressions for the bispectrum are

$$\mathbf{S_x}(e^{j\omega'},e^{j\omega}) = \mathbf{P_x}(e^{j\omega'},e^{j\omega})\,\mu_L(\omega-\omega') \tag{6}$$

$$\mathbf{P_x}(e^{j\omega'},e^{j\omega}) = \frac{1}{L}\sum_{r=0}^{L-1}\sum_{i=-\infty}^{\infty}\mathbf{p_x}(i,r)e^{-j\omega' i}e^{j\omega r} \tag{7}$$

where $\mathbf{p_x}(i,r)= E[\mathbf{x}(i)\mathbf{x}^\dagger(r)]$ and

$$\mu_L(y) = \sum_{q=-\infty}^{\infty}\delta(y+\frac{2\pi q}{L}) \tag{8}$$

The function describing the shape of the impulse along the q-th line is $\mathbf{P}^q_\mathbf{x}(e^{j\omega}) = \mathbf{P_x}(e^{j\omega},e^{j(\omega-(2\pi q/L))})$. Note that $\mathbf{P}^q_\mathbf{x}(e^{j\omega}) = \mathbf{P}^{q+L}_\mathbf{x}(e^{j\omega})$ for all integers q. Here $\mathbf{P}^0_\mathbf{x}(e^{j\omega})$ is Hermitian positive-semidefinite for all real ω. In the WSS case, $L=1$ and this quantity equals the conventional psd matrix of the WSS process. Thus the bispectrum matrix of a WSS process $\mathbf{x}(n)$ with psd matrix $\mathbf{S}(e^{j\omega})$ has a plot as shown in Fig. 3b, and is given by

$$\mathbf{S_x}(e^{j\omega'},e^{j\omega}) = \mathbf{S}(e^{j\omega})\sum_{q=-\infty}^{\infty}\delta(\omega-\omega'+2\pi q) \tag{9}$$

If ever a vector process $\mathbf{x}(n)$ has a bispectrum matrix given by (6) then it is necessarily CWSS(L). Similarly, for any $\mathbf{S_x}(e^{j\omega'},e^{j\omega})$ given by (9) for Hermitian positive semidefinite $\mathbf{S}(e^{j\omega})$ one can find a WSS process $\mathbf{x}(n)$ with $\mathbf{S}(e^{j\omega})$ as the psd matrix, and hence with $\mathbf{S_x}(e^{j\omega'},e^{j\omega})$ as the bispectrum matrix.

[1]The term bispectrum has also been used in the literature [7] to refer to third order statistics of random processes, however our definition is completely different from this usage. Our definition is similar but not identical to that of the cyclic spectrum used in [8].

2.3. Action of linear systems on Bispectra

It is well known that passing a WSS vector $\mathbf{x}(n)$ with psd matrix $\mathbf{S_x}(e^{j\omega})$ through an LTI system with frequency response $\mathbf{H}(e^{j\omega})$, gives a WSS vector $\mathbf{y}(n)$ with psd matrix

$$\mathbf{S_y}(e^{j\omega}) = \mathbf{H}(e^{j\omega})\mathbf{S_x}(e^{j\omega})\mathbf{H}^\dagger(e^{j\omega}) \tag{10}$$

Thus the output psd matrix is the transfer function of the cascade shown in Fig. 4 where each box is an LTI system with transfer function written within. Similarly, passing an arbitrary random vector process $\mathbf{x}(n)$ with bispectrum matrix $\mathbf{S_x}(e^{j\omega'},e^{j\omega})$ through an LTV system with bifrequency function $\mathbf{K}(e^{j\omega'},e^{j\omega})$ gives a vector $\mathbf{y}(n)$ which is in general nonstationary. The bispectrum matrix $\mathbf{S_y}(e^{j\omega'},e^{j\omega})$ of $\mathbf{y}(n)$ is the bifrequency function of the cascade shown in Fig. 5, where each box is an LTV system with bifrequency function written within. This is shown in [9] for continuous-time scalar systems, and can be proved for the present case in a similar way. Here in particular if the LTV system is LTI with transfer matrix $\mathbf{H}(e^{j\omega})$ then

$$\mathbf{S_y}(e^{j\omega'},e^{j\omega}) = \mathbf{H}(e^{j\omega'})\mathbf{S_x}(e^{j\omega'},e^{j\omega})\mathbf{H}^\dagger(e^{j\omega}) \tag{11}$$

This can be proved independently or by specializing the general result for LTV systems to the LTI case.

3. VECTOR INTERPOLATION FILTERS

This section states and proves one of the main results of the paper. We first define MIMO aliasfree(L) transfer matrices. Recall that an aliasfree(L) set of frequencies is a set S such that no two points ω and ω' satisfying $\omega-\omega'=(2\pi q/L)$ can simultaneously belong to S, if q is any integer not a multiple of L. A scalar LTI system is said to be aliasfree(L) (or anti-aliasing(L)) if its transfer function is supported on an aliasfree(L) set. The output of such a system can be decimated by L without causing aliasing overlap in the frequency domain.

Definition. The transfer matrix $\mathbf{H}(e^{j\omega})$ of a MIMO LTI system is said to have an aliasfree(L) support, and the system is said to be MIMO aliasfree(L), if it satisfies the following property: If $\omega-\omega'=(2\pi q/L)$ where q is any integer not a multiple of L, then at least one of the two matrices $\mathbf{H}(e^{j\omega})$ and $\mathbf{H}(e^{j\omega'})$ is zero. This is equivalent to the statement that there exists an aliasfree(L) set S such that each of the scalar transfer functions within the matrix $\mathbf{H}(e^{j\omega})$ has support contained in S.

3.1. Statement and Discussion of the Result

Theorem 1a. The vector interpolation filter shown in Fig. 2 has a WSS output $\mathbf{y}(n)$ for all WSS input $\mathbf{x}(n)$ if and only if the LTI system $\mathbf{H}(e^{j\omega})$ is MIMO aliasfree(L). Under this condition the psd matrices $\mathbf{S_x}(e^{j\omega})$ and $\mathbf{S_y}(e^{j\omega})$ of $\mathbf{x}(n)$ and $\mathbf{y}(n)$ respectively, are related as

$$\mathbf{S_y}(e^{j\omega}) = \frac{1}{L}\mathbf{H}(e^{j\omega})\mathbf{S_x}(e^{j\omega L})\mathbf{H}^\dagger(e^{j\omega}) \tag{12}$$

Theorem 1b. In Fig. 2, if the expanders are removed, the output $\mathbf{y}(n)$ is WSS **for all CWSS(L)** $\mathbf{v}(n)$ if and only if $\mathbf{H}(e^{j\omega})$ is MIMO aliasfree(L). In other words, a MIMO LTI system produces WSS outputs for all CWSS(L) inputs if and only if it is MIMO aliasfree(L).

Note that the definition of MIMO aliasfree transfer matrices, when applied to scalar systems, yields the usual definition of a system with aliasfree support. Thus the aliasfree condition derived in Theorem 4.1 of [1] is a special case of Theorem 1a for the case of scalar systems as in Fig. 1. The condition in Theorem 1a is clearly necessary for Theorem 1b because the vector process $\mathbf{v}(n)$ in Fig. 2 can be shown to be CWSS(L) for any WSS input $\mathbf{x}(n)$. However if $\mathbf{v}(n)$ is an arbitrary CWSS(L) vector process, then it cannot always be created by L-fold upsampling of a WSS process $\mathbf{x}(n)$; and then it is not obvious whether the condition of Theorem 1a would still ensure that the output $\mathbf{y}(n)$ is WSS. The strength of Theorem 1b is that it tells that this is indeed sufficient. This result is not stated in any form in [1].

The main idea in the proof of the result is that the system output is in general CWSS(L), with an impulsive bispectrum as in Fig. 3a. The impulse functions on the lines have shapes that depend on the transfer matrix $\mathbf{H}(e^{j\omega})$ in a way that can be calculated. We now impose that the output be WSS. This is thus equivalent to choosing $\mathbf{H}(e^{j\omega})$ so that the unwanted impulse-lines (i.e. the lines $\omega' - \omega = 2\pi q/L$, $q = 1, 2, \ldots, L-1$) are suppressed, and the output bispectrum looks as in Fig. 3b. The proof will show that this translates almost immediately into a condition on the support of $\mathbf{H}(e^{j\omega})$, in contrast to the approach of [1] which is much more laborious.

3.2. Proof of Theorem 1a

We can verify from the definitions, that L-fold upsampling of a process causes its bispectrum to be upsampled by L in each 'frequency' variable. From this observation and using (11), we can compute the bispectrum matrix $\mathbf{S}_{\mathbf{y}}(e^{j\omega'}, e^{j\omega})$ of the output in Fig. 2, to be

$$\frac{1}{L}\mathbf{H}(e^{j\omega'})\mathbf{S}_{\mathbf{x}}(e^{j\omega L})\mathbf{H}^{\dagger}(e^{j\omega})\mu_L(\omega - \omega'), \qquad (13)$$

where $\mathbf{S}_{\mathbf{x}}(e^{j\omega})$ is the conventional psd matrix of $\mathbf{x}(n)$. Here we have used (9) for the input bispectrum, and have also used the scaling property of the $\delta(.)$ function. Now (6) shows that $\mathbf{y}(n)$ is CWSS(L), and will be WSS if and only if the following is true:

$$\mathbf{H}(e^{j\omega'})\mathbf{S}_{\mathbf{x}}(e^{j\omega L})\mathbf{H}^{\dagger}(e^{j\omega}) = 0 \text{ if } \omega' - \omega = 2\pi q/L, \qquad (14)$$

for all $q \in B$ where B is the set of integers that are not multiples of L. (The system $\mathbf{H}(e^{j\omega})$ must suppress the unwanted impulse-lines in the CWSS bispectrum). Consider the special case of Fig. 1 where $\mathbf{H}(e^{j\omega}) = H(e^{j\omega})$ and $\mathbf{S}_{\mathbf{x}}(e^{j\omega}) = S_x(e^{j\omega})$ are scalars. This is the case addressed in [1]. Here (14) becomes

$$H(e^{j\omega'})S_x(e^{j\omega L})H^{\dagger}(e^{j\omega}) = 0 \text{ if } \omega' - \omega = 2\pi q/L, \qquad (15)$$

for all $q \in B$. If the output $y(n)$ is to be WSS *for all* WSS inputs $x(n)$, then it is necessary and sufficient that (15) holds for every positive $S_x(e^{j\omega L})$. (The necessity is because, as is well known, given any positive transfer function $S(e^{j\omega})$ we can find a WSS scalar random process with $S(e^{j\omega})$ as the psd matrix.) Clearly this is the same as saying that whenever $\omega' - \omega = 2\pi q/L$ for any integer q not a multiple of L, then of $H(e^{j\omega'})$ and $H(e^{j\omega})$ at least one is zero. This is nothing but the statement that the LTI system $H(e^{j\omega})$ has an aliasfree(L) support. This proves the scalar result (Theorem 4.1 of [1]).

For the more general vector case of Fig. 2, $\mathbf{y}(n)$ is WSS for all $\mathbf{x}(n)$ if and only if (14) holds for every Hermitian positive semidefinite matrix $\mathbf{S}_{\mathbf{x}}(e^{j\omega L})$. (Again, given any Hermitian positive semidefinite matrix $\mathbf{A}$ one can find a matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}\mathbf{B}^{\dagger}$; so by (10), if $\mathbf{e}(n)$ is a process with white uncorrelated scalar components, we can form the process $\mathbf{x}(n)$=$\mathbf{B}\mathbf{e}(n)$ which has psd matrix $\mathbf{S}_{\mathbf{x}}(e^{j\omega}) = \mathbf{A}$). From the definition it is clear that if $\mathbf{H}(e^{j\omega})$ is MIMO aliasfree(L) then (14) is indeed satisfied for every $\mathbf{S}_{\mathbf{x}}(e^{j\omega L})$. The converse is proved by contradiction, first for the case when $\mathbf{H}(e^{j\omega})$ is a row vector : Let $\omega' - \omega = 2\pi q/L$, $q \in B$. Choosing $\mathbf{S}_{\mathbf{x}}(e^{j\omega}) = \mathbf{I}$ shows that $\mathbf{H}(e^{j\omega})$, $\mathbf{H}(e^{j\omega'})$ are orthogonal. But if they are both nonzero, we can easily find a linear transform matrix $\mathbf{B}$ that transforms them into two non-orthogonal vectors. Choosing $\mathbf{S}_{\mathbf{x}}(e^{j\omega}) = \mathbf{B}\mathbf{B}^{\dagger}$ now furnishes the contradiction to the assumption that they are both nonzero. This can then be used to prove by contradiction the statement for the general case when $\mathbf{H}(e^{j\omega})$ is of arbitrary size. This gives the converse statement. Finally, under the MIMO aliasfree(L) condition, since the output $\mathbf{y}(n)$ is WSS, its bispectrum (13) takes the form of (9). Comparing these equations shows that the output psd is indeed given by (12) as claimed. This concludes the proof of Theorem 1a. $\nabla\nabla\nabla$

3.3. Proof of Theorem 1b

As explained in Section 3.1, we just need to show that the MIMO aliasfree(L) property of $\mathbf{H}(e^{j\omega})$ implies that $\mathbf{y}(n)$ is WSS for all CWSS(L) $\mathbf{v}(n)$ in Fig. 2. If $\mathbf{v}(n)$ is CWSS(L), the form of its bispectrum $\mathbf{S}_{\mathbf{v}}(e^{j\omega'}, e^{j\omega})$ is given by (6), and using (11), the output bispectrum has the form

$$\mathbf{S}_{\mathbf{y}}(e^{j\omega'}, e^{j\omega}) = \mathbf{H}(e^{j\omega'})\mathbf{P}_{\mathbf{v}}(e^{j\omega'}, e^{j\omega})\mathbf{H}^{\dagger}(e^{j\omega})\mu_L(\omega - \omega') \qquad (16)$$

Comparison with (6) shows that $\mathbf{y}(n)$ is indeed CWSS(L). The MIMO aliasfree(L) condition immediately implies that the function on the q-th line of the bispectrum of $\mathbf{y}(n)$, i.e.

$$\mathbf{H}(e^{j\omega})\mathbf{P}_{\mathbf{v}}(e^{j\omega}, e^{j(\omega - 2\pi q/L)})\mathbf{H}^{\dagger}(e^{j(\omega - 2\pi q/L)}),$$

is zero unless q is a multiple of L. Hence (16) takes the form of (9), i.e. $\mathbf{y}(n)$ is WSS as claimed. $\nabla\nabla\nabla$

4. OTHER RESULTS AND CONCLUSION

Looking at the bispectra allows us to find an explicit condition for the output of a scalar LPTV(L) system to be WSS for all WSS inputs. This is obtained by applying the rule described by Fig. 5 to LPTV systems in order to find the output bispectrum. As explained earlier, we then impose the condition that the unwanted impulse-lines in the bispectrum be suppressed. This gives rise to a condition in terms of the aliasing gains $A_q(e^{j\omega})$ of the system:

$$A_i(e^{j(\omega + \frac{2\pi(i-r)}{L})})A_r^*(e^{j\omega}) = 0 \;\; \text{whenever } i \neq r, \qquad (17)$$

for all $i, r \in \{0, 1, \ldots, L-1\}$. This can be verified to hold if the LPTV(L) system is in particular an exponential modulator or a LTI system, for which the result is known to be true [1]. It also leads to the following result :

Theorem 2. A rational LPTV(L) scalar system produces WSS outputs for all WSS inputs if and only if it is either a rational LTI system or an exponential LPTV(L) modulator or a cascade of these.

These results can be applied to other systems such as principal component reconstructions. Details can be found in [10].

We have thus shown that some problems involving multirate systems with stochastic inputs can be easily and elegantly solved by looking at the output bispectra. The main advantage is the geometric insight obtained when CWSS processes are involved, which gives a neat representation of the bispectra in terms of impulsive lines.

5. REFERENCES

[1] V.P.Sathe and P.P.Vaidyanathan, "Effects of Multirate Systems on the Statistical Properties of Random Signals," *IEEE Trans. Signal Processing,* vol. 41, no. 1, pp. 131-146, Jan. 1993.

[2] R.E.Crochiere and L.R.Rabiner, *Multirate Digital Signal Processing.* Prentice-Hall, 1983.

[3] L.A.Zadeh, "Frequency Analysis of Variable Networks," *Proc. IRE,* vol. 32, pp. 291-299, Mar. 1950.

[4] A.Papoulis, *Probability, Random Variables, and Stochastic Processes.* McGraw-Hill, 1965.

[5] P.P.Vaidyanathan, *Multirate Systems and Filter Banks.* Englewood Cliffs, NJ: Prentice-Hall, 1993.

[6] C.M.Loeffler and C.S.Burrus, "Optimal Design of Periodically Time-Varying and Multirate Digital Filters," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. ASSP-32, no. 5, pp. 991-997, Oct. 1984.

[7] C.L.Nikias and J.M.Mendel, Introduction to "Special Section on Higher Order Spectral Analysis," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. 38, no. 7, pp. 1236-1237, July 1990.

[8] W.A.Gardner, "Exploitation of spectral redundancy in cyclostationary signals," *IEEE SP Mag.,* Apr. 1991.

[9] A.Blanc-Lapierre and R.Fortet, *Theory of Random Functions.* Volumes I and II, Translated by J.Gani. Gordon and Breach Science Publishers, 1968.

[10] Sony Akkarakaran and P.P.Vaidyanathan, "Bifrequency and Bispectrum Maps : A New Look at Multirate Systems with Stochastic Inputs", in preparation.

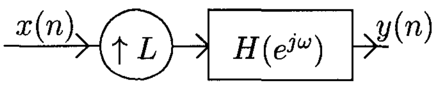

Fig. 1. Scalar interpolation filter - a special case of Fig. 2.

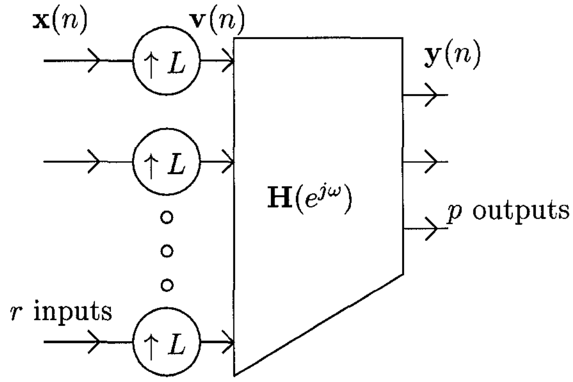

Fig. 2. A general vector interpolation filter.

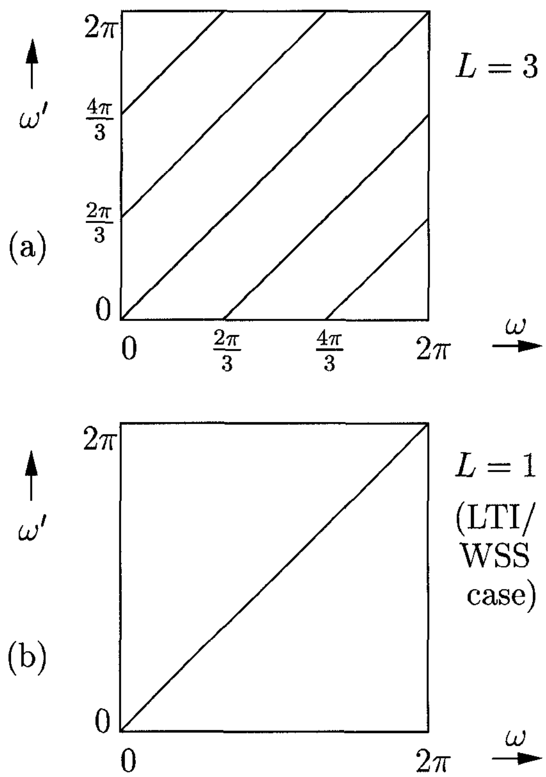

Fig. 3. Impulsive lines in LPTV(L) bifrequencies and CWSS(L) bispectra.

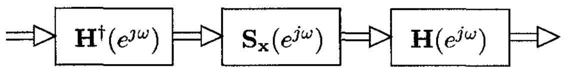

Fig. 4. Schematic explanation of the effect of an LTI system on the psd matrix.

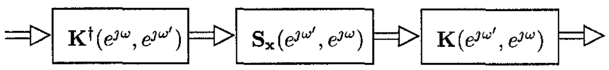

Fig. 5. Schematic explanation of the effect of an LTV system on the bispectrum matrix.