# OPTIMIZED ORTHONORMAL TRANSFORMS FOR SNR IMPROVEMENT BY SUBBAND PROCESSING

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## ABSTRACT

Orthonormal transforms for signal representation are ubiquitous in a large number of signal-processing and communications problems. A problem that frequently arises is to find the best transform for a particular situation. This paper considers the problem of noise suppression by subband processing. The noisy input is split into separate frequency bands using an orthogonal transform or filter-bank (FB), and these bands are separately denoised. We address the problem of finding the best FB for such a scheme. We have recently pointed out a strong connection between optimality of the FB and the *principal component* property, which in particular solves this problem when the noise is white. Here we establish results for the case when the noise is colored. We prove that if a *common* principal component filter-bank (PCFB) for the signal and the noise exists, it is optimal within the class of memoryless transforms. We show how this result need not be true for more general classes, using the class of all (unconstrained) FB's as an illustration.

## 1. INTRODUCTION

A general subband processing scheme using a $M$-channel uniform filter-bank (FB) is shown in Fig. 1. The FB is said to be orthonormal if the matrix $\mathbf{E}(e^{j\omega})$ is unitary for all $\omega$. Systems of this kind arise frequently in signal processing and communications. For example, the processors $P_i$ could be quantizers for signal compression, or Wiener filters for noise reduction. A closely allied system called a transmultiplexer is obtained by interchanging the analysis and synthesis sections, and arises in a communications context related to orthogonal frequency–division multiplexing.

A problem of considerable interest is that of finding the best orthonormal transform for signal representation for a particular task. For example, it has been addressed in [1, 2] when the $P_i$ of Fig. 1 are quantizers for signal compression, under the high bit-rate quantization noise models. In this case, the FB that minimizes the mean-square reconstruction error is one that minimizes the product of the variances of its subband signals. Here it is well known that within the class of transform coders (systems as in Fig. 1 where $\mathbf{E}(z)$ is a constant matrix), the best FB is the signal KLT. The best FB in the class of *all* (unconstrained) orthonormal FB's has been found in [1]. In both these cases, the optimum FB has been observed to satisfy the *principal component* property.

We have recently pointed out a stronger connection between this property and the FB optimization problem [3]. We have demonstrated [3] that *principal component filter-banks* (PCFB's) [4] are optimal whenever the objective to be minimized is a concave function of the subband variances produced by the FB. Such objectives arise in many situations, for example in the above-mentioned compression problem. The problem we focus on in this paper is one where the system of Fig. 1 takes a noisy signal as input, and aims to remove the noise. The noise is assumed to be uncorrelated to the signal, and the processors $P_i$ are either Wiener filters or hard-threshold operators (described later). For this situation, our earlier work [3] has already shown optimality of PCFB's when the noise is white. Here we consider the case where the noise is colored. We show that a *common* PCFB for the signal and the noise has certain optimality properties. For instance, it is optimal within the class of transform coders for all the above noise suppression problems. The same however need not be true for all classes of FB's, as we illustrate using the class of all (unconstrained) $M$-channel orthonormal FB's.

## 2. FORMULATION OF THE OBJECTIVE

We are given a class $\mathcal{C}$ of uniform $M$-channel orthonormal FB's, and a set of $M$ subband processors $P_i$, $i = 0, 1, \ldots, M-1$ (numbered arbitrarily). In the present paper, the $P_i$ are always constant multipliers whose values may or may not depend on the statistical properties that their inputs are assumed to have. The FB-based noise suppression system using these processors is shown in Fig. 1. The FB input is $x(n) = s(n) + \mu(n)$ where $s(n)$ is the pure desired signal and $\mu(n)$ is zero mean additive noise uncorrelated to $s(n)$. The problem at hand is to find the best FB from $\mathcal{C}$ for use in the system of Fig. 1, i.e. the FB that minimizes some measure of the reconstruction error signal $e(n) = s(n) - y(n)$. To formulate this measure, we assume that the signals $s(n)$ and $\mu(n)$ are CWSS($M$) (wide sense cyclostationary with period $M$) random processes. Thus, $e(n)$ is also CWSS($M$), and the error measure used is the variance of $e(n)$ averaged over its period of cyclostationarity $M$. We denote by $v_i^{(s)}(n)$ the $i$-th subband signal produced when the signal $s(n)$ is input to the FB,

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for $i = 0, 1, \ldots, M-1$ (where the subbands are numbered according to the subband processors they are fed into). As the FB is orthonormal, the error measure takes the form

$$\frac{1}{M}\sum_{i=0}^{M-1} E[|v_i^{(e)}|^2], \quad \text{where} \tag{1}$$

$$v_i^{(e)}(n) = v_i^{(s)}(n) - v_i^{(y)}(n), \quad \text{for } i = 0, 1, \ldots, M-1 \tag{2}$$

Thus $v_i^{(s)}(n)$ serves as the desired response that the processor $P_i$ must try to approximate at its output as best as possible in the sense of minimizing $E[|v_i^{(e)}|^2]$.

Notice that for $i = 0, 1, \ldots, M-1$, the random processes $v_i^{(e)}, v_i^{(s)}$ and $v_i^{(\mu)}$ are jointly WSS (wide sense stationary). Let $E[|v_i^{(s)}|^2] = \sigma_i^2$ and $E[|v_i^{(\mu)}|^2] = \eta_i^2$ denote the variances of $v_i^{(s)}$ and $v_i^{(\mu)}$ respectively. We consider situations where the optimization objective depends only on these variances. In particular, in our problems we have

$$E[|v_i^{(e)}(n)|^2] = f_i(\sigma_i^2, \eta_i^2) \tag{3}$$

When the processor $P_i$ is a constant multiplier $k_i$,

$$v_i^{(e)}(n) = (1-k_i)v_i^{(s)}(n) - k_i v_i^{(\mu)}(n), \quad \text{hence} \tag{4}$$

$$f_i(x, y) = |1-k_i|^2\, x + |k_i|^2\, y \tag{5}$$

When $P_i$ is a zeroth order Wiener filter or a hard-thresholder, it is a multiplier $k_i$ matched to the variances $\sigma_i^2$ and $\eta_i^2$, i.e.

$$k_i = \begin{cases} \frac{\sigma_i^2}{\sigma_i^2+\eta_i^2} & \text{if } P_i = 0^{\text{th}} \text{ order Wiener filter} \\ \left\{\begin{array}{ll} 1 & \text{if } \sigma_i^2 \geq \eta_i^2 \\ 0 & \text{otherwise} \end{array}\right\} & \text{if } P_i = \text{hard-thresholder} \end{cases}$$

Insertion in $f_i(\sigma_i^2, \eta_i^2)$ of (5) shows that (3) holds here, with

$$f_i(x, y) = \begin{cases} \frac{xy}{x+y} & \text{if } P_i = 0^{\text{th}} \text{ order Wiener filter} \\ \min(x, y) & \text{if } P_i = \text{hard thresholder} \end{cases} \tag{6}$$

We define the signal and noise **subband variance vectors** for the system of Fig. 1, as

$$\mathbf{v}_\sigma = (\sigma_0^2, \sigma_1^2, \ldots, \sigma_{M-1}^2)^T, \quad \mathbf{v}_\eta = (\eta_0^2, \eta_1^2, \ldots, \eta_{M-1}^2)^T \tag{7}$$

respectively. The set of all possible $\mathbf{v}_\sigma$ corresponding to all FB's in the given class $\mathcal{C}$ is denoted by $S_\sigma$. The set $S_\eta$ is similarly defined as the set of all possible $\mathbf{v}_\eta$. Lastly, we define the optimization **search-space** as the set $S_v$ consisting of all vectors $\mathbf{v} = (\mathbf{v}_\sigma^T, \mathbf{v}_\eta^T)^T$ such that there is a FB in $\mathcal{C}$ producing $\mathbf{v}_\sigma$ and $\mathbf{v}_\eta$ as the signal and noise subband variance vectors respectively. Thus $S_v$ is a subset of the cartesian product $S_\sigma \times S_\eta$. In the above problems, the objective can be represented purely as a function of the vector $\mathbf{v} \in S_v$. Thus the search for the best FB in $\mathcal{C}$ is equivalent to the search for the best vector in the set $S_v$. The remaining sections consider objectives of this kind. Note that we do not assume any constraint as to which processor $P_i$ to use in which subband of the FB. Thus, using the same FB (i.e. the same set of analysis and synthesis filters) we can create a new system by interchanging the coupling between its subbands and the processors. This results in a corresponding new set of subband variance vectors, which are appropriately permuted versions of the original variance vectors. Thus, the optimum vector in $S_v$ enables us to construct the optimum system of Fig. 1 by (1) proper choice of FB from $\mathcal{C}$, and (2) proper coupling of its subbands to the subband processors.

## 3. REVIEW OF PCFB OPTIMALITY

### 3.1. Relevant tools from convexity theory

**Convex sets**: A set $D \subset \mathcal{R}^M$ is said to be convex if any *convex combination* of elements of $D$ is also in $D$. A convex combination of elements of $D$ is a vector of the form $\sum_{i=1}^N \alpha_i \mathbf{d}_i$ with $\mathbf{d}_i \in D$, $\alpha_i \in [0, 1]$ and $\sum_{i=1}^N \alpha_i = 1$.
**Convex hull**: The convex hull of a set $S$ is denoted by $\text{co}(S)$ and defined as the set of all possible convex combinations of elements of $D$.
**Concave function**: A function $g$ defined on a convex set $D$ is said to be concave if its graph lies above its chord, i.e. if $f(\alpha\mathbf{d}_1 + (1-\alpha)\mathbf{d}_2) \geq \alpha f(\mathbf{d}_1) + (1-\alpha)f(\mathbf{d}_2)$ for all $\mathbf{d}_1, \mathbf{d}_2 \in D$ and $\alpha \in [0, 1]$.
**Corners (or extreme points) of convex sets**: These are defined in [5, 6]. Given a compact convex set $D \subset \mathcal{R}^M$, the set $E$ consisting of its corners is the 'smallest' (technically, *minimal*) subset of $D$ whose convex hull $\text{co}(E)$ is $D$. This fact can be used as the definition of a corner for our purposes, and it leads to an almost immediate proof [5] of the following result [6]:

**Theorem 1: Optimality of corners.** Given any function $g$ that is concave on a compact convex set $D$, at least one corner of $D$ is a minimum of $g$ over $D$. Further if $g$ is strictly concave, its minimum is necessarily at a corner of $D$.

### 3.2. Connection to PCFB optimality

**Definition of PCFB's**: Let us be given a class $\mathcal{C}$ of uniform orthonormal $M$-channel FB's, and the power-spectrum of the input to the FB. A PCFB for the class $\mathcal{C}$ is defined to be a FB that minimizes (over all FB's in $\mathcal{C}$) the mean-square error caused by dropping the $P$ weakest (lowest variance) subbands, for any $P = 0, 1, \ldots, M$. The PCFB and its existence depends on both the class $\mathcal{C}$ and the input spectrum.

To explain the optimality of PCFB's, we assume henceforward that the sets $S_\sigma, S_\eta$ and $S_v$ defined in Section 2 (and hence their convex hulls) are compact, as this is true in most physical situations. When the given class $\mathcal{C}$ of FB's contains a PCFB for the signal $s(n)$, the set $S_\sigma$ has a very special property [3]: Its convex hull $\text{co}(S_\sigma)$ is a *polytope*, i.e. its set of corners $E_\sigma$ is *finite*. Further all these corners are vectors in $S_\sigma$ that correspond to the same FB in $\mathcal{C}$, namely the PCFB. (Thus these vectors are permutations of each other.) So Theorem 1 proves that the PCFB for $s(n)$ is optimal whenever the objective to be minimized is a function that is concave on the set $\text{co}(S_\sigma)$. Situations where such objectives arise are discussed in [3].

## 4. RESULTS FOR COLORED NOISE

We now address PCFB optimality assuming that the objective function to be minimized is concave on the set

$$T \stackrel{\triangle}{=} \text{co}(S_\sigma) \times \text{co}(S_\eta) = \text{co}(S_\sigma \times S_\eta) \supset co(S_v) \tag{8}$$

With $\mathbf{v}_\sigma, \mathbf{v}_\eta$ as in (7), all objectives described by

$$f(\mathbf{v}) = \frac{1}{M}\sum_{i=0}^{M-1} f_i(\sigma_i^2, \eta_i^2) \quad \text{for } \mathbf{v} = (\mathbf{v}_\sigma^T, \mathbf{v}_\eta^T)^T \in S_v \quad (9)$$

satisfy this condition whenever the $f_i$ are concave on the non-negative quadrant of $\mathcal{R}^2$. For our denoising problems of Section 2, (1) and (3) show that the objectives are indeed described by (9). Further the $f_i$ of (5), (6) can indeed be verified to be concave. Thus, all results of the present section apply to these problems. In particular when $\mu(n)$ is white, then $S_\eta$ consists of exactly one element, and the objective can be described as a function that is concave on the set $\text{co}(S_\sigma)$. Thus, from Section 3.2, a PCFB for the signal $s(n)$ is always optimal (if it exists). Notice that such a PCFB is also a *common* PCFB for $s(n)$ and $\mu(n)$, since any FB is a PCFB for a white input. Such a common PCFB (if it exists) has certain optimality properties even if neither $s(n)$ nor $\mu(n)$ is white. These properties can no longer be explained by Section 3.2, and are stated and proved below.

### 4.1. Statement of results

**Result 1.** If the power spectrum (psd) matrices of the $M$-fold blocked versions of the signal and noise are scaled versions of each other, then for any class $\mathcal{C}$, a common signal and noise PCFB (if it exists) is always optimal.

**Result 2.** If $\mathcal{C}$ is the class of all $M$-channel orthogonal transform coders, a common signal and noise PCFB is always optimal no matter what the psd matrices corresponding to $s(n)$ and $\mu(n)$ are (provided of course they are such that a common PCFB exists). Notice that in this case, the common PCFB is also a PCFB for the signal $x(n) = s(n) + \mu(n)$. This is not true in general, for example for the class of all (unconstrained) $M$-channel FB's.

**Result 3.** If $\mathcal{C}$ is the class of all $M$-channel orthonormal FB's (unconstrained), there are large classes of psd matrices of $s(n)$ and $\mu(n)$ for which a common signal and noise PCFB exists but is still not optimal for many concave objectives.

**Result 4.** As Result 3 shows, for arbitrary class $\mathcal{C}$ and input psd matrices, a common signal and noise PCFB is not necessarily optimal for *all* concave objectives. However, it is still always optimal for a certain nontrivial subset of these objectives. There is a simple finite procedure that decides whether or not a given concave objective falls in this subset.

### 4.2. Proofs of results

Result 1 is true because under the condition it imposes, the signal and noise subband variance vectors for any FB are obtainable from each other by a constant scaling. Thus, the objective can be rewritten so that it depends on only one of these subband variance vectors, and we can use the result of Section 3.2. Notice that in this case, a PCFB for the signal is also a PCFB for the noise and vice-verca. To prove the remaining results, we assume at the outset that a separate PCFB exists for both the signal $s(n)$ and the noise $\mu(n)$ (since this is a condition in all these results). As discussed in Section 3.2, this implies $\text{co}(S_\sigma) = \text{co}(E_\sigma)$ where $E_\sigma \subset S_\sigma$ is a finite set of vectors all of which correspond to the signal PCFB (and are hence permutations of each other). Similarly $\text{co}(S_\eta) = \text{co}(E_\eta)$ where $E_\eta \subset S_\eta$ is a finite set corresponding to the noise PCFB. Thus $T = \text{co}(E_\sigma) \times \text{co}(E_\eta) = \text{co}(E_\sigma \times E_\eta)$. So it can be seen that $T$ is a polytope whose set of corners is the (finite) Cartesian product $E_\sigma \times E_\eta$. Thus, $S_v \subset \text{co}(S_v) \subset T$, where $T$ is a polytope.

All corners of $\text{co}(S_v)$ lie in $S_v$, and at least one of them is the optimum (Theorem 1). So we try to examine the nature of the corners of $\text{co}(S_v)$. The first observation is that any corner of the polytope $T$ that lies in $S_v$ is also a corner of $\text{co}(S_v)$. To establish results 2,3,4 of Section 4.1, we now assume that a common signal and noise PCFB exists. Now any vector in $S_v$ that corresponds to such a PCFB is a corner of $T$. Conversely, any corner of $T$ that lies in $S_v$ corresponds to such a PCFB. Let $E_v \subseteq E_\sigma \times E_\eta$ denote the set of all corners of $T$ that lie in $S_v$. The points in $E_v$ are hence corners of $\text{co}(S_v)$, but it is however not clear whether or not $\text{co}(S_v)$ has other corners. In the extreme situation when $E_v = E_\sigma \times E_\eta$, then in fact $\text{co}(S_v) = T$ and it does not have other corners. Since all its corners correspond to a common signal and noise PCFB, such a PCFB would always be optimal. However, $E_v = E_\sigma \times E_\eta$ is usually possible only in contrived cases, or with a degeneracy such as white noise.

**Proof of Result 2**

If $\mathcal{C}$ is the class of $M$-channel orthogonal transform coders, the common signal and noise PCFB is the common KLT, which is unique. So all vectors in $E_v$ are corners of $\text{co}(S_v)$ that correspond to the common KLT, and it turns out that $\text{co}(S_v)$ *has no other corners.* Thus, $\text{co}(S_v) = \text{co}(E_v)$ is a polytope with set of corners $E_v$, and at least one of these corners minimizes the objective (Theorem 1). So a common signal and noise PCFB (KLT) is always optimal. The crucial fact that $\text{co}(S_v) = \text{co}(E_v)$ is true no matter what the signal and noise spectra are (assuming of course that they are such that a common KLT exists). It is shown in detail in [7], and thus completes the proof.

**Proof of Result 3**

When $\mathcal{C}$ is the class of all $M$-channel orthonormal FB's (unconstrained) then the property $\text{co}(S_v) = \text{co}(E_v)$ still holds in certain restricted situations, for example when the signal and noise psd matrices are both constant. (In this case the PCFB's are the corresponding KLT's.) However, it does not hold for all signal and noise psd matrices. If it does not hold, it implies that $\text{co}(S_v)$ has other corners besides the points in $E_v$. There would then be concave objectives for which one of these other corners (which clearly does not correspond to a common signal and noise PCFB) is optimal. This is illustrated by the example of Fig. 2. Here the filter-bank FB-II is not a PCFB for either the signal or the noise. However it is better than FB-I, the common signal and noise PCFB for $\mathcal{C}$, for the denoising problem using either hard-thresholding or zeroth order Wiener filtering in both subbands. This can be verified by substituting the corresponding signal and noise subband variance vectors from Fig. 2 into the objectives for these problems. Note that many more such examples can be created, for instance by applying small perturbations on the spectra in Fig. 2. This proves Result 3 of Section 4.1. Fig. 3 shows the various geometries of $S_v$ as a subset of $T$ arising out of the situations discussed thus far. (The figure is only illustrative, since $T$

actually lies in an even dimensional space and not in $\mathcal{R}^3$.)

**Proof of Result 4**

We know from Theorem 1 that at least one of the finitely many corners of $T$ is a minimum of the objective *over* $T$. Such corners, though easy to find, may not lie in $S_v$, and are hence not useful in general since we are seeking for minima over $S_v$ (or $\text{co}(S_v)$). However, there are always concave objectives with the property that their minimum over $T$ is a corner of $T$ that actually lies in $S_v$. This corner would hence be the minimum of such an objective over $S_v \subset T$ too. Thus for the subset $\mathcal{F}$ of concave objectives having this property, a common PCFB is optimal. ($\mathcal{F}$ is however not the complete set of concave objectives for which a common signal and noise PCFB is optimal.) Recall that the (finite) set $E_\sigma \times E_\eta$ of corners of $T$ is fully specified given the signal and noise subband variance vectors generated by the common signal and noise PCFB. Thus we have a simple finite procedure to identify whether or not a given concave objective is in the subset $\mathcal{F}$: We evaluate the objective at each corner of $T$, and find whether or not the minimum over these corners (which is the minimum over $T$) lies in $S_v$. This establishes Result 4 of Section 4.1.

## 5. CONCLUSION

We have used the tools of [3, 5] to prove results on PCFB optimality for certain colored noise suppression problems. We have shown optimality of the common signal and noise PCFB (KLT) within the class of all orthogonal transform coders, for subband noise suppression by zeroth order Wiener filtering or by hard-thresholding. We have shown that for other classes of FB's, even if a common signal and noise PCFB exists it will not be optimal in general. Some extensions to biorthogonal FB's, and further details and examples can be found in [7].

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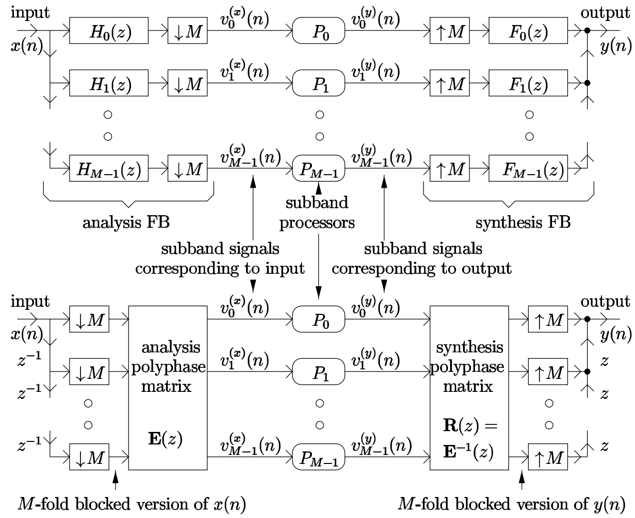


Figure 1: Generic FB based signal processing scheme.

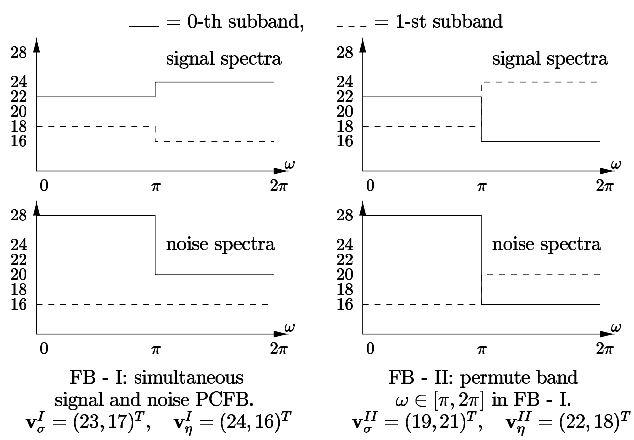


Figure 2: Suboptimality of signal and noise PCFB.

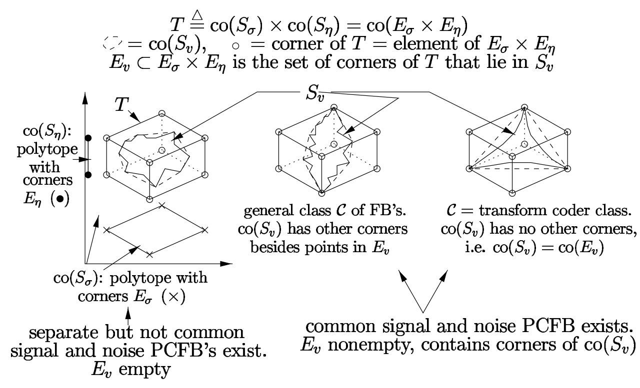


Figure 3: Geometry of the search-space.