Bifrequency and Bispectrum Maps: A New Look at Multirate Systems with Stochastic Inputs

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***Abstract*—In multirate digital signal processing, we often encounter decimators, interpolators, and complicated interconnections of these with LTI filters. We also encounter cyclo-wide-sense stationary (CWSS) processes and linear periodically time-varying (LPTV) systems. It is often necessary to understand the effects of multirate systems on the statistical properties of their input signals. Some of these issues have been addressed earlier. For example, it has been shown that a necessary and sufficient condition for the output of an L-fold interpolation filter to be wide sense stationary (WSS) for all WSS inputs is that the filter have an alias-free (L) support. However, several questions of this nature remain unanswered. For example, what is the necessary and sufficient condition on a pair (or more generally a *bank*) of interpolation filters so that their outputs are *jointly* WSS (JWSS) for all jointly WSS inputs? What is the condition if only *the sum* of their outputs is required to be WSS? When is the output of an LPTV system (for example a uniform filter-bank) WSS for all WSS inputs? Some of these questions may appear to be simple generalizations of the above-mentioned result for a single interpolation filter. However, the frequency domain approaches that proved this result are quite difficult to generalize to answer these questions. The purpose of this paper is to provide these answers using analysis based on bifrequency maps and bispectra. These tools are two-dimensional (2-D) Fourier transforms that characterize all linear time-varying (LTV) systems and nonstationary random processes, respectively. We show that the questions raised above can be addressed elegantly and in a geometrically insightful way using these tools. We also derive a bifrequency characterization of lossless LTV systems. This may potentially lead to an increased understanding of these systems.**

I. INTRODUCTION

MULTIRATE systems contain interconnections of decimators, interpolators, and LTI filters. Linear periodically time-varying (LPTV) systems and cyclo-wide-sense stationary (CWSS) random processes occur frequently in multirate processing [1], [7], [13], [15]. It is often required to analyze the effects of multirate systems on the statistics of their input. Some analysis of this kind has been carried out in [1], where a necessary and sufficient condition is derived for the output of an L-fold interpolation filter to be WSS for all WSS inputs. The condition is that the filter should have an alias-free(L) support. However, many questions remain unanswered. For example, what is the generalization of this condition for the case of multi-input multi-output (MIMO) systems with WSS vector inputs? What is the condition if we have a general LPTV system instead of the interpolation filter?

This paper addresses issues of this kind. We show that the alias-free (L) condition mentioned above, as well as many of the other results of [1], can be obtained in an elegant and geometrically insightful manner using bifrequency and bispectrum analysis. The bifrequency map [2], [3] gives a complete description of a general linear time-varying (LTV) system. For nonstationary vector random processes, the autocorrelation matrix is a function of two indices. Its two-dimensional (2-D) Fourier transform, which we shall call the bispectrum matrix (or simply bispectrum for scalar processes) gives a complete description of the second-order statistics of the process. These tools have not often been used to analyze multirate systems because they are sometimes too general for the purpose. However, they greatly simplify the analysis of the issues raised above and in the abstract. Thus, the bifrequency and bispectrum "domain" is the natural domain for addressing questions of this nature. The analysis of [1] based on pseudocirculant power spectral density (psd) matrices would prove to be inordinately complicated for this purpose. We also point out a necessary and sufficient bifrequency characterization of the lossless LTV systems described in [5] and [6]. The condition is somewhat more general than that of [5] and [6] and may potentially give additional insights into these systems.

A. Previous Work

For a general continuous-time nonstationary scalar random process, the autocorrelation function depends on two "time" variables. Many properties of its 2-D Fourier transform can be found in [4] and [8]. These 2-D Fourier transforms are repeatedly referred to in this paper and are called *bispectra* for convenience. The term bispectrum has also been used in the literature on higher order spectral analysis [9] to denote the 2-D Fourier transform of the *third-order statistics* of the random process. Thus, this second definition is totally different from what we mean here, and we will make no further reference to works based on it. The *bifrequency* function for general scalar LTV systems has been defined in [3] for continuous-time and in [2] for discrete-time systems. Bifrequency maps are used in [10] for design of multirate filters and LPTV systems. They are used in [2, ch. 3] to obtain beautiful geometric insights into the operation of basic multirate building blocks with deterministic inputs; however, in this case, the results could also be obtained using other methods such as polyphase matrices. Stochastic inputs have, however, not been considered in [2].

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CWSS processes arise naturally in our analysis. Such processes have been observed and studied by a number of authors. For example, Gardner discusses continuous-time CWSS processes in signal processing and communications applications [11], [12] using a tool called the cyclic spectrum, which is somewhat different from the bispectrum. The cyclic spectral density matrix is used for discrete time CWSS processes by Ohno and Sakai in [13] and [14], and several results have been established. If the CWSS(M) process is passed through M modulators providing frequency shifts of $2\pi k/M$, $k = 0, 1, \cdots, M-1$ and the results are then passed separately through ideal lowpass filters of bandwidth $2\pi/M$, then the vector consisting of the M resulting outputs is WSS [14], and the cyclic spectral density matrix is the psd matrix of this vector. The M-fold *blocked version* (Section II) of the CWSS(M) scalar process is also a WSS vector process, and its psd matrix is related to the cyclic spectral density matrix by the Gladyshev's relation [14].

In [14], the cyclic spectral density matrix is computed for (discrete time) periodic AR processes and for the output of a filterbank. It is shown that if the FB is alias-free, its output is WSS for all WSS inputs. The cyclic spectrum is used in [13] to numerically optimize filterbanks to minimize the reconstruction error after some subbands are dropped. In [15] and [16], Petersohn *et al.* have presented a matrix calculus description of multirate systems. It is used to compute the spectra of output signal and noise in systems such as cascaded multirate filters and fractional decimation circuits. It is also used to derive an efficient polyphase structure for fractional decimation. These earlier works have not considered more complex situations involving *vector* CWSS processes. They have also not considered the conditions for stationarity of the output of more complicated systems like vector interpolation filters. While the present paper was in the final stages of preparation, the very recent reference [17] also came to our attention. This reference deals with the properties of higher order spectra in the context of multirate processing.

B. Outline of the Paper

Section II provides a review of the basic definitions and properties of stationary and cyclostationary discrete random processes. Section III is a review of the basic properties of bifrequency maps and bispectra, which we will need for our analysis. Section IV examines the effect of elementary multirate building blocks such as decimators and expanders on the bispectra of their random process inputs. These results are then used in the later sections to analyze more complicated multirate systems. Section V considers vector interpolation filters, which upsample the input vector process and pass the result through a MIMO transfer matrix. We find the necessary and sufficient condition on this transfer matrix so that the output is WSS for all WSS inputs. Section VI considers general LPTV scalar systems. In particular, we show that the only rational LPTV systems that produce WSS output for all WSS inputs are rational LTI systems, exponential LPTV modulators, and cascades of these. These results are applied to other multirate systems such as principal component filterbanks in Section VII. We also point out the bifrequency characterization of lossless LTV systems described in [5] and [6].

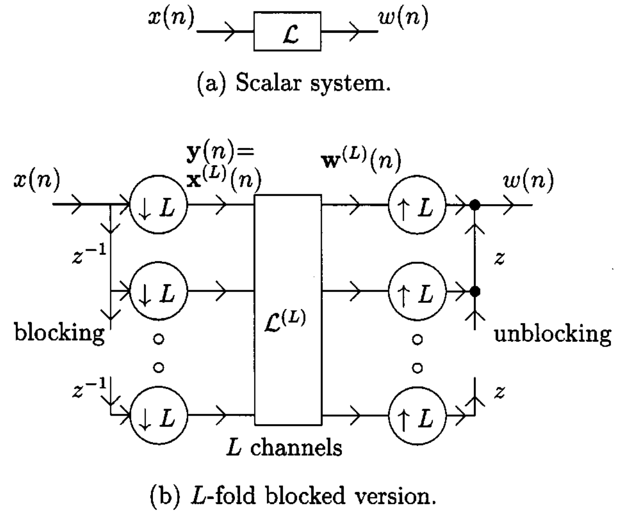

Fig. 1. Blocking.

II. NOTATIONS AND PRELIMINARIES

A. Notations

Superscripts $(^*)$ and $(^T)$ denote the complex conjugate and matrix (or vector) transpose, respectively, whereas superscript dagger $(^\dagger)$ denotes the conjugate transpose. Boldface letters are used for matrices and vectors. The (i, j) element of a matrix $\mathbf{B}$ is denoted by $[\mathbf{B}]_{(i,j)}$. Lower-case letters are used for 1-D and 2-D discrete sequences, whereas upper-case letters are used for 1-D and 2-D Fourier transforms. $\mathcal{Z}$ and $\mathcal{R}$, respectively, denote the set of integers and that of real numbers. The space of all finite norm M-component vector sequences is denoted by $l^2(M)$. [The l^2 norm of a vector sequence $\mathbf{x}(n)$ is defined as $\|\mathbf{x}(n)\| = [\sum_n \mathbf{x}^\dagger(n)\mathbf{x}(n)]^{1/2}$.] The DFT matrix of order L is denoted by $\mathbf{W}_L$. Decimators, expanders, and other multirate building blocks have their standard definitions and symbols in figures, which can be found, for example, in [18].

B. Preliminaries

Multirate systems contain decimators and expanders in addition to LTI systems. Therefore, their study involves the study of linear periodically time-varying (LPTV) systems and "blocked versions" of scalar systems. Stationary random processes, when passed through LPTV systems, become cyclostationary. Since these ideas occur frequently later, we begin by defining them. A central theme of this paper is to study the effect of multirate systems on the statistics of random process inputs. All random processes are assumed to be zero mean since the effect of a linear system on the mean can easily be analyzed.

1) Blocking: Fig. 1(a) shows a scalar linear system $\mathcal{L}$ with input $x(n)$ and output $w(n)$. Fig. 1(b) shows the L-fold blocked version of this system. The vector $\mathbf{x}^{(L)}(n) = [x(nL), x(nL-1), \cdots, x(nL-L+1)]^T$ is said to be the L-fold blocked version of $x(n)$, and similarly, $\mathbf{w}^{(L)}(n)$ is the blocked version of the output $w(n)$. Conversely, $x(n)$ is called the L-fold unblocked version of $\mathbf{x}^{(L)}(n)$. The L-input L-output system $\mathcal{L}^{(L)}$ of Fig. 1(b) is said to be the L-fold blocked version of the scalar system $\mathcal{L}$ of Fig. 1(a). If $\mathcal{L}$ is linear, so is $\mathcal{L}^{(L)}$. Further, $\mathcal{L}$ is LTI if and only if $\mathcal{L}^{(L)}$ is LTI with a pseudocirculant transfer matrix (defined in [1]). $\mathcal{L}$ is LPTV(L) (defined in Section III-B) iff $\mathcal{L}^{(L)}$ is LTI.

2) Cyclostationarity: Given a vector random process $\mathbf{x}(n)$, define its autocorrelation sequence as $\mathbf{r_x}(n,m) = E[\mathbf{x}(m)\mathbf{x}^\dagger(m-n)]$. If this is periodic in m with period L for all integers n, we say that $\mathbf{x}(n)$ is CWSS(L), i.e. wide sense cyclostationary with period L. If $L=1$, then $\mathbf{x}(n)$ is wide sense stationary (WSS), and $\mathbf{r_x}(n,m) = \mathbf{r_x}(n)$ is independent of m. In this case, the z-transform of $\mathbf{r_x}(n)$ is called the power spectrum (psd) matrix $\mathbf{S_x}(z)$ of the process.

3) Joint Cyclostationarity: Two vector processes $\mathbf{x}(n)$ and $\mathbf{y}(n)$ are said to be jointly CWSS(L) [JCWSS(L)] if the vector $\begin{bmatrix}\mathbf{x}(n)\\ \mathbf{y}(n)\end{bmatrix}$ is CWSS(L). It can be shown that a vector process is CWSS(L) iff all pairs of its component scalar processes are JCWSS(L). If $L=1$, JCWSS(L) is synonymous with jointly WSS (JWSS).

4) Blocking of CWSS Processes: Let $x(n), y(n)$ be scalar random processes with respective L-fold blocked versions $\mathbf{x}^{(L)}(n)$ and $\mathbf{y}^{(L)}(n)$. Then, $x(n)$ is CWSS(L) iff $\mathbf{x}^{(L)}(n)$ is WSS. In addition, $x(n)$ is WSS iff $\mathbf{x}^{(L)}(n)$ is WSS with pseudocirculant psd matrix. Last, $x(n), y(n)$ are JCWSS(L) iff $\mathbf{x}^{(L)}(n)$ and $\mathbf{y}^{(L)}(n)$ are JWSS.

To motivate how processes with properties as defined above appear in multirate systems, note, for example, that upsampling a WSS vector process (i.e., upsampling each component) by L gives a CWSS(L) vector process. Another example is a multistage implementation of an interpolation filter, i.e., a repeated cascade of an expander and a filter. This kind of cascade also occurs in nonuniform tree-structured filterbanks. This gives rise to processes that are CWSS with larger and larger periods.

III. Basic Properties of Bifrequencies and Bispectra

In order to describe the time-varying systems and nonstationary processes that are invariably encountered in the study of multirate systems, we now review the bifrequency and bispectrum descriptions.

A. General LTV Systems and Bifrequency Maps

A MIMO LTV system [2] with input $\mathbf{x}(n)$ and output $\mathbf{y}(n)$ is fully specified by the time-domain relation

$$\mathbf{y}(m) = \sum_{n=-\infty}^{\infty} \mathbf{k}(m,n)\mathbf{x}(n) = \sum_{n=-\infty}^{\infty} \mathbf{h}(m,n)\mathbf{x}(m-n). \tag{1}$$

Here, $\mathbf{k}(m,n)$ is called the *Green's function* and is perfectly general. The function $\mathbf{h}(m,n)$ is the time-varying impulse response that is useful only if the input and output rates are equal [2]. These are related as $\mathbf{h}(m,n) = \mathbf{k}(m,m-n)$. The LTV system is also fully specified by the bifrequency function

$$\mathbf{K}(e^{j\omega'}, e^{j\omega}) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathbf{k}(m,n)e^{-j\omega' m}e^{j\omega n}. \tag{2}$$

The system input–output relation in the frequency domain is

$$\mathbf{Y}(e^{j\omega'}) = \int_{-\pi}^{\pi} \mathbf{K}(e^{j\omega'}, e^{j\omega})\mathbf{X}(e^{j\omega})\, d\omega. \tag{3}$$

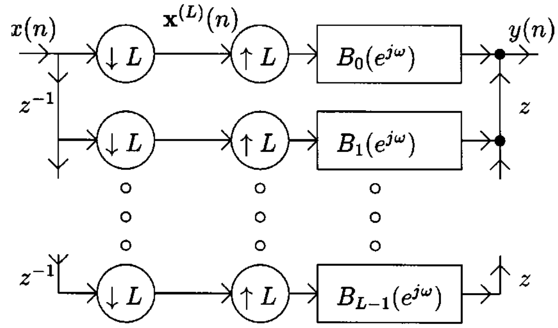

Fig. 2. General representation of scalar LPTV(L) systems.

An example of an LTV system is the (scalar) modulator defined by the input-output relation $y(n) = a_0(n)x(n)$. It has a bifrequency map $K(e^{j\omega'}, e^{j\omega}) = (1/2\pi)A_0(e^{j(\omega'-\omega)})$ [2], where $A_0(e^{j\omega})$ is the Fourier transform of $a_0(n)$. A generalization is what may be called a "rational" LTV system, i.e., one that is realizable by a linear difference equation with time-varying coefficients. Thus, it is characterized by $y(n) = a_0(n)x(n) + a_1(n)x(n-1) + \cdots + a_k(n)x(n-k)$ and can be shown to have bifrequency map

$$K(e^{j\omega'}, e^{j\omega}) = \frac{1}{2\pi} \sum_{i=0}^{k} e^{-j\omega i}\, A_i(e^{j\omega'-\omega)}) \tag{4}$$

where $A_i(e^{j\omega})$ is the Fourier transform of $a_i(n)$ for $i = 0, 1, \cdots, k$. This expression clearly brings out the fact that the system is characterized by the $k+1$ transfer functions $A_i(e^{j\omega})$.

Cascading two LTV systems with Green's functions $\mathbf{k}_i(m,n)$ and bifrequencies $\mathbf{K}_i(e^{j\omega'}, e^{j\omega})$, $i = 1, 2$ (in that order) gives a new LTV system with Green's function $\mathbf{k}(m,n)$ and bifrequency $\mathbf{K}(e^{j\omega'}, e^{j\omega})$ given by

$$\mathbf{k}(m,n) = \sum_{r=-\infty}^{\infty} \mathbf{k}_2(m,r)\mathbf{k}_1(r,n) \tag{5}$$

$$\mathbf{K}(e^{j\omega'}, e^{j\omega}) = \int_{-\pi}^{\pi} \mathbf{K}_2(e^{j\omega'}, e^{j\omega''})\mathbf{K}_1(e^{j\omega''}, e^{j\omega})\, d\omega''. \tag{6}$$

B. LPTV Systems

An LPTV(L) system (linear periodically time-varying with period L) is defined as one whose impulse response $h(m,n)$ is periodic in m with period L for each integer n. By LPTV system, we will always mean one with equal input and output rates so that the impulse response $h(m,n)$ can be meaningfully used. This paper considers only scalar LPTV(L) systems. Such systems can always be represented as in Fig. 2 (see [18, ch. 10]), and conversely, the system of Fig. 2 is always LPTV(L). In this figure, the boxes are scalar LTI systems with transfer functions $B_r(z)$ and impulse responses $b_r(n)$. Using standard multirate tools, we can show that $b_r(n) = h(n-r,n)$ for $r = 0, 1, \cdots, L-1$ for all integers n. The system is characterized by the L transfer functions $B_r(z)$. It can also be viewed as a maximally decimated filterbank (in which the L analysis filters are delays). Hence, it can alternatively be characterized by the

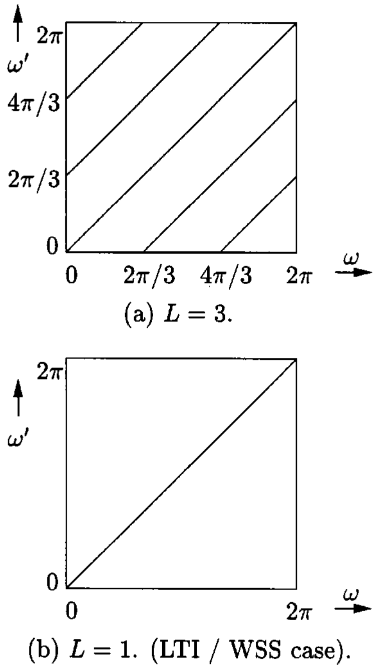

Fig. 3. Impulsive lines in LPTV(L) bifrequencies and CWSS(L) bispectra. (a) $L = 3$. (b) $L = 1$ (LTI/WSS case).

L "aliasing gains" $A_r(z)$ [18] of the filterbank, which describe its input-output relation according to

$$Y(e^{j\omega}) = \sum_{q=0}^{L-1} A_q(e^{j\omega})X(e^{j(\omega-(2\pi q/L))}). \tag{7}$$

Applying the relations in [18, ch. 5] to the system in Fig. 2 shows that

$$\begin{aligned}&[A_0(z), \quad A_1(z), \quad \cdots, \quad A_{L-1}(z)]^T \\ &\quad = \frac{1}{L}\,\mathbf{W}_L^{\dagger}[B_0(z), \quad B_1(z), \quad \cdots, \quad B_{L-1}(z)]^T. \end{aligned}\tag{8}$$

From (7), we can expect the bifrequency function $K(e^{j\omega'}, e^{j\omega})$ for this system to be characterized in some way by L transfer functions. Indeed, we have [10]

$$K(e^{j\omega'}, e^{j\omega}) = F(e^{j\omega'}, e^{j\omega}) \sum_{q=-\infty}^{\infty} \delta\left(\omega - \omega' + \frac{2\pi q}{L}\right) \tag{9}$$

where

$$F(e^{j\omega'}, e^{j\omega}) = \frac{1}{L}\sum_{r=0}^{L-1}\sum_{i=-\infty}^{\infty} k(i,r)e^{-j\omega' i}e^{j\omega r} \tag{10}$$

and where $k(m,n) =$ Green's function. Equation (9) shows that the bifrequency map consists of a set of parallel impulsive lines, as illustrated by Fig. 3(a). The qth line has equation $\omega' - \omega = 2\pi q/L$. The shape of the impulse along this line is given by the transfer function $F_q(e^{j\omega}) = F(e^{j\omega}, e^{j(\omega-(2\pi q/L))})$, which is thus seen to have an L-fold periodicity in q. The bifrequency map is fully characterized by the first L of the functions $F_q(e^{j\omega})$, i.e., we have

$$\begin{aligned}F_q(e^{j\omega}) &= A_q(e^{j\omega}), \quad q = 0, 1, \cdots, L-1; \quad \text{and} \\ F_q(e^{j\omega}) &= F_{q+L}(e^{j\omega}) \quad \text{for all integers } q.\end{aligned}\tag{11}$$

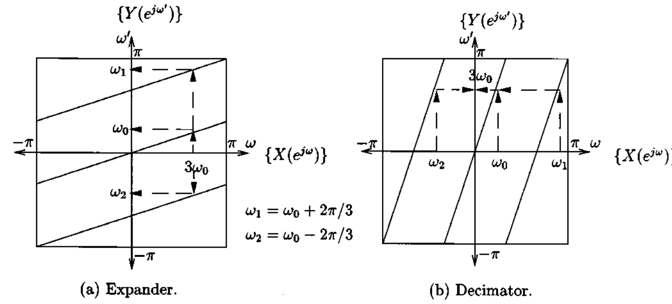

Fig. 4. Bifrequency maps of three-fold expander and decimator. (a) Expander. (b) Decimator.

To prove this, we insert (9) in (3) and integrate using the sifting property of the $\delta(\cdot)$ function, which yields

$$\begin{aligned}Y(e^{j\omega'}) &= \sum_{q=0}^{L-1} F(e^{j\omega'}, e^{j(\omega'-(2\pi q/L))})X(e^{j(\omega'-(2\pi q/L))}) \\ &= \sum_{q=0}^{L-1} F_q(e^{j\omega'})X(e^{j(\omega'-(2\pi q/L))}).\end{aligned}\tag{12}$$

Comparison with (7) now proves (11).[1]

Conversely, for any function $K(e^{j\omega'}, e^{j\omega})$ given by (9), we can find an LPTV(L) system with bifrequency function $K(e^{j\omega'}, e^{j\omega})$. This is because by inverse Fourier transformation, we see that the corresponding impulse response $h(m,n)$ is periodic in m with period L for all integers n, which is in agreement with the definition of an LPTV(L) system. If the system is in fact LTI with transfer function $H(e^{j\omega})$, then $L = 1$ and $F(e^{j\omega}, e^{j\omega}) = H(e^{j\omega}) = F_q(e^{j\omega})$ for all integers q.

Expanders and decimators are also LTV systems but are not LPTV under our definition as their input and output rates are not equal. Their bifrequencies also consist of parallel impulsive lines [2], but unlike the case of LPTV systems, the slope of these lines is not unity. It is less than unity for expanders and greater than unity for decimators [2]. These bifrequency maps are shown in Fig. 4, which is a pictorial representation of the frequency domain input-output behavior (3) of these systems. It shows exactly how the expander creates imaging and the decimator creates aliasing in the frequency domain, for deterministic inputs.

C. General Nonstationary Processes and Bispectra

The autocorrelation matrix $\mathbf{r_x}(m,n)$ and bispectrum matrix $\mathbf{S_x}(e^{j\omega'}, e^{j\omega})$ of a nonstationary vector process $\mathbf{x}(n)$ are defined as

$$\mathbf{r_x}(m,n) = E[\mathbf{x}(m)\mathbf{x}^{\dagger}(n)] \tag{13}$$

$$\mathbf{S_x}(e^{j\omega'}, e^{j\omega}) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \mathbf{r_x}(m,n)e^{-j\omega' m}e^{j\omega n}. \tag{14}$$

When $\mathbf{x}(n)$ is a scalar, $\mathbf{S_x}(e^{j\omega'}, e^{j\omega}) = S_x(e^{j\omega'}, e^{j\omega})$ is a scalar as well and satisfies the following properties: (These are dis-

[1]We can also prove (11) in the time domain by using (8), (10), and the relations $h(m,n) = k(m, m-n)$ and $b_r(n) = h(n-r, n)$.

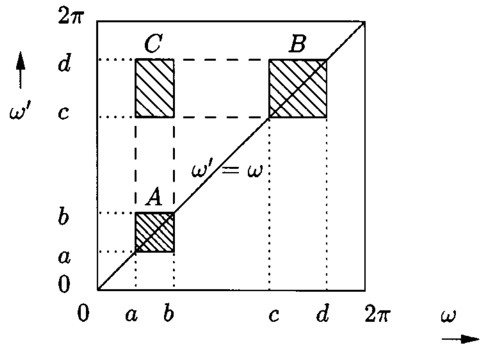

Fig. 5. Illustrating properties of scalar bispectra.

crete-time versions of the properties stated in [4] for continuous-time processes.)

$$S_x(e^{j\omega}, e^{j\omega}) \geq 0 \quad \text{for all real } \omega \tag{15}$$

$$\int_a^b \int_a^b S_x(e^{j\omega'}, e^{j\omega}) d\omega' d\omega \geq 0 \tag{16}$$

$$\int_a^b \int_a^b S_x(e^{j\omega'}, e^{j\omega})\, d\omega'\, d\omega \int_c^d \int_c^d S_x(e^{j\omega'}, e^{j\omega})\, d\omega'\, d\omega \geq \left| \int_a^b \int_c^d S_x(e^{j\omega'}, e^{j\omega})\, d\omega'\, d\omega \right|^2. \tag{17}$$

Here, $a, b, c, d \in (-\pi, \pi]$. Fig. 5 illustrates these properties. Equation (15) says that the 2-D function $S_x(e^{j\omega'}, e^{j\omega})$ is non-negative on the diagonal $\omega' = \omega$, and (16) says that its integrals over the hatched areas A,B are non-negative. To help understand these properties, we can draw analogies between the bispectrum $S_x(e^{j\omega'}, e^{j\omega})$ and the conventional power spectrum (psd) matrix $\mathbf{P}(e^{j\omega})$ of a WSS vector process. Thus, we have the following.

- Equation (15) is analogous to $[\mathbf{P}(e^{j\omega})]_{(ii)} \geq 0$ for all real ω.
- Equation (16) is analogous to the positive semidefiniteness property $\mathbf{x}^\dagger \mathbf{P}(e^{j\omega})\mathbf{x} \geq 0$ for all vectors $\mathbf{x}$.
- Equation (17) is analogous to the property $|\mathbf{x}^\dagger \mathbf{P}(e^{j\omega})\mathbf{y}|^2 \leq (\mathbf{x}^\dagger \mathbf{P}(e^{j\omega})\mathbf{x})(\mathbf{y}^\dagger \mathbf{P}(e^{j\omega})\mathbf{y})$ (for any vectors $\mathbf{x}, \mathbf{y}$), which follows from the Cauchy–Schwartz inequality.
- Although the diagonal elements of $\mathbf{P}(e^{j\omega})$ are non-negative functions, the off-diagonal elements need not even be real. Similarly, although $S_x(e^{j\omega}, e^{j\omega}) \geq 0$ (15), $S_x(e^{j\omega'}, e^{j\omega})$ need not even be real if $\omega \neq \omega'$.

In general, when $\mathbf{x}(n)$ is not a scalar, $\mathbf{S}_\mathbf{x}(e^{j\omega}, e^{j\omega})$ is Hermitian positive semidefinite for all real ω. The Hermitian property follows from (14). The positive semidefiniteness can be deduced by using (15) on the (scalar) bispectra of the scalar processes $\mathbf{v}^\dagger \mathbf{x}(n)$ for arbitrary constant vectors $\mathbf{v}$.

D. Action of Linear Systems on Bispectra

It is well known that when a WSS vector $\mathbf{x}(n)$ with psd matrix $\mathbf{S}_\mathbf{x}(e^{j\omega})$ is passed through an LTI system with frequency response $\mathbf{H}(e^{j\omega})$, the output is a WSS vector $\mathbf{y}(n)$ with psd matrix

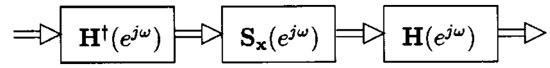

Fig. 6. Schematic explanation of the effect of an LTI system on the psd matrix.

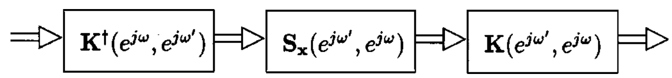

Fig. 7. Schematic explanation of the effect of an LTV system on the bispectrum matrix.

$$\mathbf{S}_\mathbf{y}(e^{j\omega}) = \mathbf{H}(e^{j\omega})\mathbf{S}_\mathbf{x}(e^{j\omega})\mathbf{H}^\dagger(e^{j\omega}). \tag{18}$$

Thus, the output psd matrix is the transfer function of the cascade shown in Fig. 6, where each box is an LTI system with the indicated transfer function. When an arbitrary random vector process $\mathbf{x}(n)$ with bispectrum matrix $\mathbf{S}_\mathbf{x}(e^{j\omega'}, e^{j\omega})$ is passed through an LTV system with bifrequency function $\mathbf{K}(e^{j\omega'}, e^{j\omega})$, the output vector $\mathbf{y}(n)$ is, in general, nonstationary. Its bispectrum matrix $\mathbf{S}_\mathbf{y}(e^{j\omega'}, e^{j\omega})$ is the bifrequency function of the cascade shown in Fig. 7, where each box is an LTV system with indicated bifrequency function. This is shown in [8] for continuous-time scalar systems and is proved for discrete-time MIMO systems in Appendix A, for completeness. In particular if the LTV system is LTI with transfer matrix $\mathbf{H}(e^{j\omega})$, then

$$\mathbf{S}_\mathbf{y}(e^{j\omega'}, e^{j\omega}) = \mathbf{H}(e^{j\omega'})\mathbf{S}_\mathbf{x}(e^{j\omega'}, e^{j\omega})\mathbf{H}^\dagger(e^{j\omega}). \tag{19}$$

This can be proved independently or by specializing the general result for LTV systems to the LTI case.

E. Bispectra of CWSS Processes

Analogous to the case of LPTV(L) systems, we can show that a CWSS(L) random vector process $\mathbf{x}(n)$ has bispectrum matrix $\mathbf{S}_\mathbf{x}(e^{j\omega'}, e^{j\omega})$ consisting of a set of parallel impulsive lines as in Fig. 3(a). The bispectrum is given by expressions analogous to (9) and (10), namely

$$\mathbf{S}_\mathbf{x}(e^{j\omega'}, e^{j\omega}) = \mathbf{P}_\mathbf{x}(e^{j\omega'}, e^{j\omega}) \sum_{q=-\infty}^{\infty} \delta\left(\omega - \omega' + \frac{2\pi q}{L}\right) \tag{20}$$

where

$$\mathbf{P}_\mathbf{x}(e^{j\omega'}, e^{j\omega}) = \frac{1}{L} \sum_{r=0}^{L-1} \sum_{i=-\infty}^{\infty} \mathbf{p}_\mathbf{x}(i, r) e^{-j\omega' i} e^{j\omega r}$$
$$\text{where } \mathbf{p}_\mathbf{x}(i, r) = E[\mathbf{x}(i)\mathbf{x}^\dagger(r)]. \tag{21}$$

The function describing the shape of the impulse along the qth line (i.e., the line $\omega' - \omega = 2\pi q/L$) is $\mathbf{P}_\mathbf{x}^q(e^{j\omega}) = \mathbf{P}_\mathbf{x}(e^{j\omega}, e^{j(\omega-(2\pi q/L))})$. Note that $\mathbf{P}_\mathbf{x}^q(e^{j\omega}) = \mathbf{P}_\mathbf{x}^{q+L}(e^{j\omega})$ for all integers q. We may separate the L component functions

$\mathbf{P}_{\mathbf{x}}^{i}(e^{j\omega}), i = 0, 1, \cdots, L-1$ that characterize the bispectrum and rewrite (20) as

$$\mathbf{S}_{\mathbf{x}}(e^{j\omega'}, e^{j\omega}) = \sum_{i=0}^{L-1} \mathbf{P}_{\mathbf{x}}^{i}(e^{j\omega'}) \cdot \sum_{m=-\infty}^{\infty} \delta\left(\omega - \omega' + \frac{2\pi i}{L} + 2\pi m\right). \quad (22)$$

From the discussion following (17), $\mathbf{P}_{\mathbf{x}}^{0}(e^{j\omega})$ is Hermitian positive-semidefinite for all real ω. In the special case when $\mathbf{x}(n)$ is WSS, $L = 1$, and $\mathbf{P}_{\mathbf{x}}^{0}(e^{j\omega})$ equals the conventional psd matrix of $\mathbf{x}(n)$. Thus, for a WSS process $\mathbf{x}(n)$ with psd matrix $\mathbf{S}(e^{j\omega})$, the bispectrum matrix has a plot as shown in Fig. 3(b) and is given by

$$\mathbf{S}_{\mathbf{x}}(e^{j\omega'}, e^{j\omega}) = \mathbf{S}(e^{j\omega}) \sum_{q=-\infty}^{\infty} \delta(\omega - \omega' + 2\pi q). \quad (23)$$

IV. Analysis of Basic Multirate Building Blocks

This section examines the effect of basic blocks such as decimators and expanders on the bispectrum of their stochastic input. Some of the results will be used in the later sections to analyze more complicated systems. Note that decimating/upsampling of a vector means performing that operation on each of its components.

A. Expanders, Decimators, and the Blocking Mechanism

1) Expanders: Let $\mathbf{z}(n)$ be a vector process obtained by L-fold upsampling of the process $\mathbf{x}(n)$, i.e.,

$$\mathbf{z}(n) = \begin{cases} \mathbf{x}(n/L), & \text{whenever } n/L \text{ is an integer} \\ 0, & \text{otherwise.} \end{cases}$$

Then, we conclude that $\mathbf{z}(n)$ has autocorrelation sequence $E[\mathbf{z}(m)\mathbf{z}^{\dagger}(n)]$ that is obtained by upsampling the autocorrelation sequence of $\mathbf{x}(n)$ by the diagonal matrix $\mathbf{L} = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}$. Thus, if $\mathbf{S}_{\mathbf{x}}(e^{j\omega'}, e^{j\omega})$ and $\mathbf{S}_{\mathbf{z}}(e^{j\omega'}, e^{j\omega})$ are, respectively, the bispectrum matrices of $\mathbf{x}(n)$ and $\mathbf{z}(n)$, we have

$$E[\mathbf{z}(m)\mathbf{z}^{\dagger}(n)] = \begin{cases} E[\mathbf{x}(m/L)\mathbf{x}^{\dagger}(n/L)] \\ \quad \text{whenever } (m/L), (n/L) \text{ are both integers} \\ 0, \quad \text{otherwise} \end{cases} \quad (24)$$

$$\mathbf{S}_{\mathbf{z}}(e^{j\omega'}, e^{j\omega}) = \mathbf{S}_{\mathbf{x}}(e^{j\omega' L}, e^{j\omega L}). \quad (25)$$

It is well known (see Fig. 4(a) [2]) that an expander creates imaging in the frequency domain for deterministic inputs. Relation (25) shows that it also creates imaging in the *bispectrum domain* for *arbitrary stochastic inputs* (not necessarily WSS). This is shown in Fig. 8. This observation will be very useful in later sections.

2) Decimators: Let the vector process $\mathbf{y}(n)$ be obtained by decimating $\mathbf{x}(n)$ by L, i.e., $\mathbf{y}(n) = \mathbf{x}(Ln)$. Then, the autocorrelation sequence of $\mathbf{y}(n)$ is obtained by decimating that of $\mathbf{x}(n)$, i.e.,

$$E[\mathbf{y}(m)\mathbf{y}^{\dagger}(n)] = E[\mathbf{x}(Lm)\mathbf{x}^{\dagger}(Ln)]. \quad (26)$$

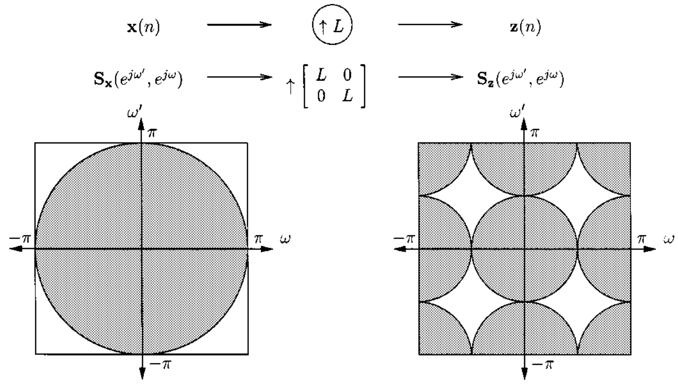

Fig. 8. Effect of an expander on the bispectrum of a nonstationary process.

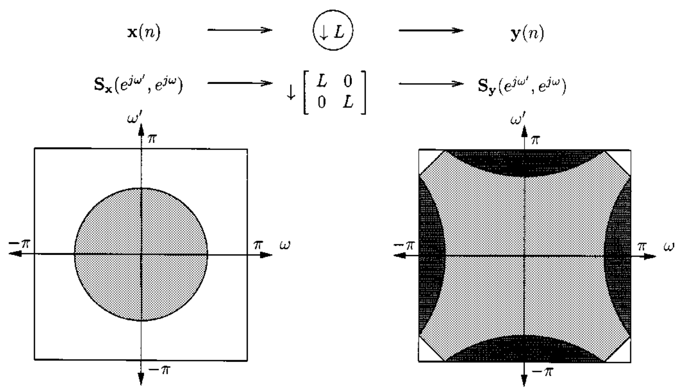

Fig. 9. Effect of a decimator on the bispectrum of a nonstationary process.

Hence, the bispectrum matrices of $\mathbf{x}(n)$ and $\mathbf{y}(n)$ are related as

$$\mathbf{S}_{\mathbf{y}}(e^{j\omega'}, e^{j\omega}) = \frac{1}{L^2} \sum_{p=0}^{L-1} \sum_{q=0}^{L-1} \mathbf{S}_{\mathbf{x}}(e^{j((\omega'-2\pi p)/L)}, e^{j((\omega-2\pi q)/L)}). \quad (27)$$

Thus, the decimator creates aliasing in the *bispectrum domain* (Fig. 9) for *stochastic inputs* (possibly nonstationary), just as it creates aliasing in the frequency domain (Fig. 4(b) [2]) for deterministic inputs. In Fig. 9, the light shade represents the region of support of the original bispectrum and that of its stretched-out version (after passage through the expander). The dark–shaded areas in the output bispectrum represent overlap with shifted copies of the stretched version.

3) Blocking: Consider a scalar process $x(n)$ and its L-fold blocked version $\mathbf{y}(n) = \mathbf{x}^{(L)}(n)$. These are related as in Fig. 1. Thus, using (19), (25), and (27), we find that their bispectra are related as

$$[\mathbf{S}_{\mathbf{y}}(e^{j\omega'}, e^{j\omega})]_{(m,n)} = \sum_{p=0}^{L-1} \sum_{q=0}^{L-1} \mathbf{S}_{\mathbf{x}}(e^{j((\omega'-2\pi p)/L)}, e^{j((\omega-2\pi q)/L)}) \cdot \exp\left(-j\left(\frac{\omega' - 2\pi p}{L}\right)m\right) \cdot \exp\left(j\left(\frac{\omega - 2\pi q}{L}\right)n\right) \quad (28)$$

and

$$\mathbf{S_x}(e^{j\omega'}, e^{j\omega}) = \sum_{m=0}^{L-1}\sum_{n=0}^{L-1} [\mathbf{S_y}(e^{j\omega' L}, e^{j\omega L})]_{(m,n)} \cdot \exp(j\omega' m)\ \exp(-j\omega n). \quad (29)$$

These equations can be used to prove certain results on blocking of CWSS processes (see Section IV-B).

B. Preliminary Results

Many of the more elementary results of [1], some of which are stated in Section II, can now be easily proved from the above discussion. Further, this proof technique generalizes these results directly to the case of vector inputs, unlike the techniques of [1].

- L-fold upsampling of a WSS vector process gives a CWSS(L) vector process. To see this, note that the bispectrum of the WSS process has impulse lines separated by a vertical spacing of 2π [see Fig. 3(b)]. Due to the bispectrum domain imaging created by the expander (Fig. 8), this spacing is "compressed" to $2\pi/L$. Therefore, the output has a CWSS(L) bispectrum as in Fig. 3(a). Note that the expander output cannot be CWSS(K) for $K < L$ (unless the input is identically zero), i.e., L is the "fundamental period" of cyclostationarity of the output.
- M-fold decimation of a CWSS(L) vector process $\mathbf{x}(n)$ gives a process $\mathbf{y}(n)$ that, in general, is CWSS(K), where $K = L/\text{gcd}(L, M)$. To prove this, note that the input bispectrum is as in Fig. 3(a), with impulses along the lines $\omega - \omega' + (2\pi i/L) = 0,\ i \in \mathcal{Z}$. Due to the bispectrum domain aliasing created by the decimator [which is shown in Fig. 9 and by (27)], the decimator output has impulses along the lines

$$\frac{\omega - \omega' + 2\pi(p-q)}{M} + \frac{2\pi i}{L} = 0, \quad \text{or equivalently} \quad (30)$$

$$\omega - \omega' + \frac{2\pi[(p-q)K + iN]}{K} = 0 \quad \text{for } i \in \mathcal{Z}, p, q \in \{0, 1, \cdots, M-1\} \quad (31)$$

  where $N = M/\text{gcd}(L, M)$. These lines certainly form a subset of the set of lines $\omega - \omega' + (2\pi j/K) = 0,\ j \in \mathcal{Z}$ in a CWSS(K) bispectrum, and hence, the output is CWSS(K).
- A scalar process is CWSS(L) iff its L-fold blocked version is a WSS vector. More generally, a scalar process is CWSS(KL) iff its L-fold blocked version is CWSS(K). This can be shown from (28) and (29).

V. Vector Interpolation Filters

This and the remaining sections analyze more complicated interconnections of the basic multirate building blocks of the last section. We derive necessary and sufficient conditions for their outputs to be WSS for all WSS inputs. The central theme in these analyses is that the multirate system fed with WSS input creates CWSS output by somehow adding more lines in the bispectrum (which is usually due to the presence of an expander). The aim is to *find the conditions under which the extra lines can be suppressed*. The geometric insights obtained by looking at the bispectra are exploited to find the conditions elegantly.

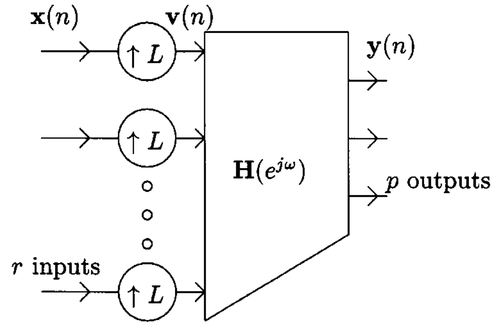

Fig. 10. General vector interpolation filter.

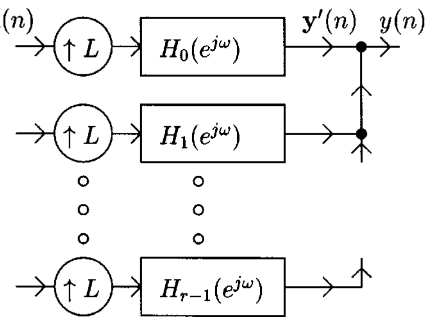

Fig. 11. Synthesis filter bank—a special case of Fig. 10.

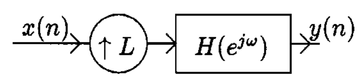

Fig. 12. Scalar interpolation filter—a special case of Fig. 10.

This section examines the L-fold vector interpolation filter, which is shown in Fig. 10. This system upsamples the r-component input vector $\mathbf{x}(n)$ by L and passes the result through a MIMO LTI system with $p \times r$ transfer matrix $\mathbf{H}(e^{j\omega})$. In general, p, r and L could be arbitrary positive integers unrelated to each other ($L > 1$). Fig. 11 shows the synthesis section of an r channel uniform filterbank with upsampling factor L. This is a special vector interpolation filter where $p = 1$, i.e., $\mathbf{H}(e^{j\omega})$ is a row vector. If the vector $\mathbf{y}'(n)$ is considered to be output in Fig. 11, we get another special case where $p = r$, and $\mathbf{H}(e^{j\omega})$ is square and diagonal. Finally, if $p = r = 1$ in Fig. 10, we get the usual scalar interpolation filter of Fig. 12.

For the special case of the scalar interpolation filter, [1] shows that the output $y(n)$ is WSS for all WSS inputs $x(n)$ if and only if the LTI filter $H(e^{j\omega})$ has an alias-free (L) support. The proof is based on the fact that a scalar process is WSS iff its blocked versions are WSS with pseudocirculant psd matrices. This proof is quite involved and does not give any indication about the corresponding result for the general system of Fig. 10. This section provides a greatly simplified proof of this result using bispectrum analysis. We show that the new proof extends without much additional effort to the general case of Fig. 10.

A. MIMO Alias-free (L) Systems

In order to state the main result on vector interpolation filters, we need to define MIMO alias-free (L) transfer matrices.

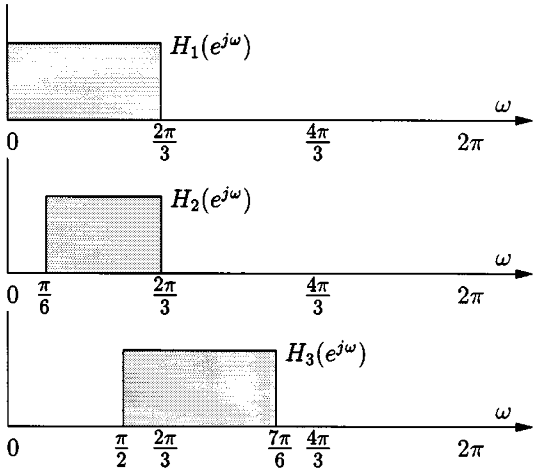

Fig. 13. Illustrating MIMO alias-free (L) supports.

Recall that an alias-free (L) set of frequencies is a set S such that no two points ω and ω' satisfying $\omega - \omega' = (2\pi q/L)$ can simultaneously belong to S for any integer q that is not a multiple of L. A scalar LTI system is defined to be alias-free (L) [or anti-aliasing (L)] if its frequency response is supported on an alias-free (L) set [1]. We now generalize this definition to MIMO LTI systems.

Definition: The transfer matrix $\mathbf{H}(e^{j\omega})$ of a MIMO LTI system is said to have an alias-free (L) support, and the system is said to be MIMO alias-free (L) if it satisfies the following property: If $\omega, \omega' \in \mathcal{R}$ such that $\omega - \omega' = (2\pi q/L)$, where q is any integer not a multiple of L, then at least one of the two matrices $\mathbf{H}(e^{j\omega})$ and $\mathbf{H}(e^{j\omega'})$ is zero. This can be seen to be equivalent to the following: There exists an alias-free (L) set S such that each of the scalar transfer functions within the matrix $\mathbf{H}(e^{j\omega})$ has support contained in S. [In particular, for example, they all could have the same support S, which is alias-free (L).]

Notice that for scalar $\mathbf{H}(e^{j\omega})$, the above reduces to the usual definition of alias-free (L) scalar systems [1]. In addition, another equivalent definition is that $\mathbf{H}(e^{j\omega})$ is an ideal "image suppressor" for deterministic inputs coming from the output of an L-fold expander. To explain image suppression for MIMO systems, note that if the vector $\mathbf{X}(e^{j\omega})$ is upsampled L-fold, the output is $\mathbf{Y}(e^{j\omega}) = \mathbf{X}(e^{j\omega L})$, each of whose components has L copies of the frequency response of the corresponding component of $\mathbf{X}(e^{j\omega})$. A MIMO alias-free (L) system acting on $\mathbf{Y}(e^{j\omega})$ would retain only one copy of each of the components and process these copies. Note that the *same* copy is retained for each component of the vector so that all these copies lie in the same frequency bands. This is illustrated in Fig. 13, where the shaded areas show the filter supports. In this figure, the matrix $[H_1(e^{j\omega}), H_2(e^{j\omega})]$ is MIMO alias-free (3), whereas $[H_1(e^{j\omega}), H_3(e^{j\omega})]$ is not MIMO alias-free (L) for any integer L, although all the individual scalar systems $H_i(e^{j\omega}), i = 1, 2, 3$ have alias-free (3) supports.

B. Statement and Implications of the Main Result

Theorem 1a: The vector interpolation filter shown in Fig. 10 has a WSS output $\mathbf{y}(n)$ for all WSS inputs $\mathbf{x}(n)$ if and only if the LTI system $\mathbf{H}(e^{j\omega})$ is MIMO alias-free (L). Under this condition, the psd matrices $\mathbf{S_x}(e^{j\omega})$ and $\mathbf{S_y}(e^{j\omega})$ of $\mathbf{x}(n)$ and $\mathbf{y}(n)$, respectively, are related as

$$\mathbf{S_y}(e^{j\omega}) = \frac{1}{L}\,\mathbf{H}(e^{j\omega})\mathbf{S_x}(e^{j\omega L})\mathbf{H}^{\dagger}(e^{j\omega}). \tag{32}$$

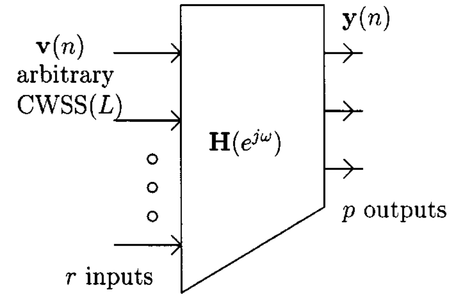

Fig. 14. Setup for Theorem 1b.

Theorem 1b: In Fig. 14, the output $\mathbf{y}(n)$ is WSS *for all CWSS(L)* $\mathbf{v}(n)$ if and only if $\mathbf{H}(e^{j\omega})$ is MIMO alias-free (L). In other words, a MIMO LTI system produces WSS outputs for all CWSS(L) inputs if and only if it is MIMO alias-free (L).

We now discuss some implications of the above results.

1) *Generalization of* [1]: We get [1, th. 4.1] from Theorem 1(a) above if $\mathbf{x}(n)$ and $\mathbf{H}(e^{j\omega})$ are scalars.
2) *Connection Between Theorems 1a and 1b*: In Fig. 10, the vector process $\mathbf{v}(n)$ is always CWSS(L) for any WSS $\mathbf{x}(n)$, as shown in Section IV. Therefore, the condition in Theorem 1a is obviously a necessary condition in Theorem 1b. However, if $\mathbf{v}(n)$ is an arbitrary CWSS(L) vector process as in Fig. 14, it cannot always be created as in Fig. 10 by upsampling a WSS process $\mathbf{x}(n)$ by L. It is then not obvious whether the condition of Theorem 1a is still sufficient for $\mathbf{y}(n)$ of Fig. 14 to be WSS. The strength of Theorem 1b is that it states that this is indeed the case. This result is not stated in any form in [1].
3) *Synthesis Filterbank*: Theorem 1a can be applied to the special vector interpolation filter of Fig. 11. With $\mathbf{x}(n)$ as input and $y(n)$ as output, $\mathbf{H}(e^{j\omega}) = [H_0(e^{j\omega}), H_1(e^{j\omega}), \cdots, H_{r-1}(e^{j\omega})]$. If the output is considered to be $\mathbf{y}'(n)$ instead of $y(n)$, then $\mathbf{H}(e^{j\omega})$ is diagonal with the ith diagonal entry equal to $H_i(e^{j\omega}), i = 0, 1, \cdots, r-1$. However, Theorem 1a says that in both cases, the necessary and sufficient condition for the output to be WSS for all WSS inputs $\mathbf{x}(n)$ is the same, i.e., that all the filters $H_i(e^{j\omega}), i = 0, 1, \cdots, r-1$ have supports contained in an aliasfree (L) set. In addition, under this condition, from Theorem 1b, the output will be WSS even if the expanders are removed and the input $\mathbf{x}(n)$ is allowed to be an arbitrary CWSS(L) process. The sum of the components of a WSS vector process is a WSS scalar process. Therefore, obviously the condition for $\mathbf{y}'(n)$ to be WSS is sufficient for $y(n)$ to be WSS, but Theorem 1 further tells us the not-so-obvious fact that it is also necessary.
4) *Relation to Perfect Reconstruction Filterbanks*: Fig. 11 is in fact the synthesis section of the uniform filterbank shown in Fig. 15. If this filterbank has the perfect reconstruction (PR) property, i.e., an input-output relation

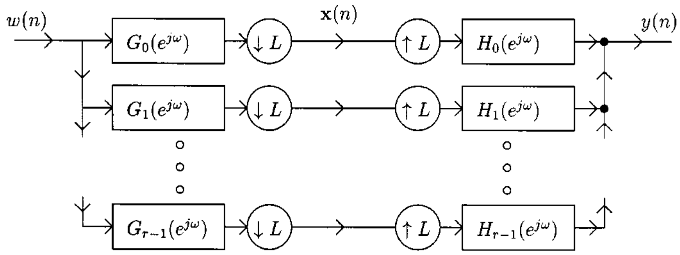

Fig. 15. General uniform filter bank.

$y(n) = cw(n-n_0)$, then clearly, the output $y(n)$ is WSS for all WSS inputs $w(n)$. However, in this case, the synthesis filters $H_i(e^{j\omega})$ cannot satisfy the MIMO alias-free condition of Theorem 1 because that would imply that all output sequences $y(n)$ have Fourier transform $Y(e^{j\omega})$ with an alias-free (L) support (violating the PR property). This apparent conflict with Theorem 1 is resolved by noting that it is only the *filterbank input* $w(n)$ that is allowed to be arbitrary. The input vector $\mathbf{x}(n)$ to the *synthesis section* of the filterbank in Fig. 15 is *not* an arbitrary WSS vector process as required by Theorem 1; it is constrained by the analysis section of the PR filterbank. The nature of this constraint is analyzed in detail in Appendix B. A question arising from this is the following: What are the conditions under which $y(n)$ is WSS for all WSS $w(n)$ in Fig. 15? This is answered in Section VI. In addition, note that for PR, it is necessary that $r \geq L$. The case where $r < L$ cannot give PR and is considered in Section VII.

5) *Joint Stationarity Properties*: Theorem 1 allows us to answer questions on joint stationarity. For example, we can obtain the condition on a pair of L-fold scalar interpolation filters for their outputs to be jointly stationary for all pairs of jointly stationary inputs. This situation is equivalent to that of Fig. 11 with $r = 2$. From [1], a necessary condition is that each filter have a support contained in an alias-free (L) set. Theorem 1 goes further to give the following necessary and sufficient condition: *Both* filter supports must be contained in some *common* alias-free (L) set. Similarly, from Theorem 1a, we can further find the necessary and sufficient condition on a pair of *vector* interpolation filters for their outputs to be jointly WSS for all jointly WSS input pairs: All the component scalars in *both* the filter transfer matrices should have supports contained in some common alias-free (L) set.

C. Proof of Theorem 1

We first prove Theorem 1a. We use (19) and (25) to compute the output bispectrum matrix in Fig. 10, as

$$\mathbf{S_y}(e^{j\omega'}, e^{j\omega}) = \frac{1}{L}\, H(e^{j\omega'})\mathbf{S_x}(e^{j\omega L})\mathbf{H}^{\dagger}(e^{j\omega}) \cdot \sum_{q=-\infty}^{\infty} \delta\left(\omega - \omega' + \frac{2\pi q}{L}\right) \tag{33}$$

where $S_x(e^{j\omega})$ is the usual psd matrix of $\mathbf{x}(n)$. Here, we have used (23) for the input bispectrum, and the scaling property of the $\delta(\cdot)$ function. Now, (20) shows that $\mathbf{y}(n)$ is CWSS(L) and will be WSS if and only if

$$\mathbf{H}(e^{j\omega'})\mathbf{S_x}(e^{j\omega L})\mathbf{H}^{\dagger}(e^{j\omega}) = 0 \quad \text{whenever} \quad \omega' - \omega = 2\pi q/L, \quad \text{for all } q \in B \tag{34}$$

where B is the set of integers that are not multiples of L. [The system $\mathbf{H}(e^{j\omega})$ must suppress the unwanted impulse-lines in the CWSS bispectrum to get a WSS bispectrum].

1) The Sathe–Vaidyanathan Special Case: Consider the special case of Fig. 12, where $\mathbf{H}(e^{j\omega}) = H(e^{j\omega})$, and $\mathbf{S_x}(e^{j\omega}) = S_x(e^{j\omega})$ are scalars. This is the case addressed in [1]. Here, (34) becomes

$$H(e^{j\omega'})S_x(e^{j\omega L})H^*(e^{j\omega}) = 0 \quad \text{whenever} \quad \omega' - \omega = 2\pi q/L, \quad \text{for all } q \in B. \tag{35}$$

The output $y(n)$ will be WSS *for all* WSS inputs $x(n)$ iff (35) holds for every non-negative $S_x(e^{j\omega L})$. [This condition is necessary because as is well known, for any transfer function $S(e^{j\omega}) \geq 0$, we can find a WSS scalar random process with psd $S(e^{j\omega})$.] Clearly, this is the same as saying that whenever $\omega' - \omega = 2\pi q/L$ for any integer q not a multiple of L, then of $H(e^{j\omega'})$ and $H(e^{j\omega})$, at least one is zero. This is precisely the statement that the LTI system $H(e^{j\omega})$ has an alias-free (L) support. This proves the scalar result (see [1, th. 4.1]).

For the more general vector case of Fig. 10, $\mathbf{y}(n)$ is WSS for all WSS $\mathbf{x}(n)$ iff (34) holds for every Hermitian positive semidefinite matrix $\mathbf{S_x}(e^{j\omega L})$. (Again, for any Hermitian positive semidefinite $\mathbf{A}$, we can find a matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}\mathbf{B}^{\dagger}$. Therefore, by (18), if $\mathbf{e}(n)$ is a process with white uncorrelated scalar components, we can form the process $\mathbf{x}(n) = \mathbf{B}\mathbf{e}(n)$ with psd matrix $\mathbf{S_x}(e^{j\omega}) = \mathbf{A}$.) From the definition of MIMO alias-free (L) systems, it is clear that if $\mathbf{H}(e^{j\omega})$ is MIMO alias-free (L), then (34) indeed holds for every $\mathbf{S_x}(e^{j\omega L})$. The lemma in Appendix C shows that if (34) holds for every Hermitian positive definite $\mathbf{S_x}(e^{j\omega L})$, then of $\mathbf{H}(e^{j\omega})$ and $\mathbf{H}(e^{j\omega'})$, at least one is the zero matrix, i.e., $\mathbf{H}(e^{j\omega})$ is MIMO alias-free (L). This gives the converse. Finally, under the MIMO alias-free (L) condition, since the output $\mathbf{y}(n)$ is WSS, its bispectrum (33) takes the form of (23). Comparing these equations shows that the psd of $\mathbf{y}(n)$ is indeed given by (32), as claimed. Notice that the lemma of Appendix C is not needed for the scalar case because it is trivial there. $\triangledown\triangledown\triangledown$

We now prove Theorem 1b. For this, it suffices to show that the MIMO alias-free (L) property of $\mathbf{H}(e^{j\omega})$ implies that $\mathbf{y}(n)$ is WSS for all CWSS(L) $\mathbf{v}(n)$ in Fig. 14. If $\mathbf{v}(n)$ is CWSS(L), the form of its bispectrum $\mathbf{S_v}(e^{j\omega'}, e^{j\omega})$ is given by (20). Therefore, using (19), the output bispectrum has the form

$$\mathbf{S_y}(e^{j\omega'}, e^{j\omega}) = \mathbf{H}(e^{j\omega'})\mathbf{P_v}(e^{j\omega'}, e^{j\omega})\mathbf{H}^{\dagger}(e^{j\omega}) \cdot \sum_{q=-\infty}^{\infty} \delta\left(\omega - \omega' + \frac{2\pi q}{L}\right). \tag{36}$$

Comparison with (20) shows that $\mathbf{y}(n)$ is CWSS(L). The MIMO alias-free (L) condition implies that

$$\mathbf{H}(e^{j\omega})\mathbf{P_v}(e^{j\omega}, e^{j(\omega - 2\pi q/L)})\mathbf{H}^{\dagger}(e^{j(\omega - 2\pi q/L)})$$

is zero unless q is a multiple of L. Hence, (36) takes the form of (23), i.e., $\mathbf{y}(n)$ is WSS. $\triangledown\triangledown\triangledown$

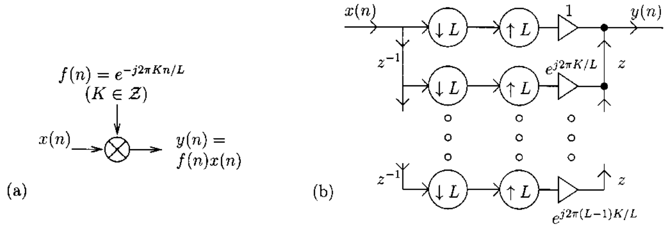

Fig. 16. Exponential LPTV(L) modulator. (a) Symbol. (b) Representation as in Fig. 2.

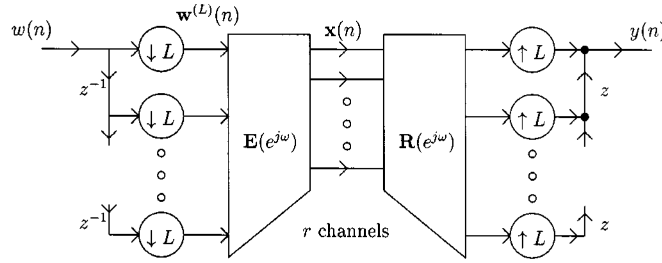

Fig. 17. Polyphase representation of a general uniform filter bank.

Thus, the MIMO alias-free (L) $\mathbf{H}(e^{j\omega})$ suppresses the unwanted lines in the bispectrum in both Theorems 1a and 1b. The only difference is that the functions on the lines were more constrained in the case of Theorem 1a; however, we showed that this does not enable us to relax the requirement on $\mathbf{H}(e^{j\omega})$ for its output to be WSS for all WSS inputs. The proof of the scalar and vector cases of Theorem 1a are almost equally easy, and the generalization to Theorem 1b is almost immediate. The approach of [1] would be inordinately complicated for these purposes.

2) Further Generalizations: The method of proof also allows us, if we so desire, to obtain more relaxed conditions, such as the conditions for the output to be CWSS(L') (rather than WSS) for all WSS inputs, where L' is any divisor of L. This requirement would again translate into a condition on the supports of the elements of the transfer matrix $\mathbf{H}(e^{j\omega})$; however, it would be less restrictive than the MIMO alias-free (L) condition. Looking at bispectra shows exactly how the requirements translate into conditions on the supports.

VI. Action of LPTV Systems on WSS Inputs

We know that an LTI system produces WSS output for all WSS inputs. Exponential modulators described by Fig. 16 also have this property, as shown in [1]. Both these systems are special cases of a general LPTV(L) scalar system shown in Fig. 2. The question that arises is whether there are other LPTV systems that have this property. This section answers this question completely, i.e., we derive a necessary and sufficient condition for an LPTV(L) system to produce WSS outputs for all WSS inputs. Using the derived condition, we show that the only *rational* LPTV systems (systems as in Fig. 2 with the filters $B_i(z)$ all rational) with this property are rational LTI systems, exponential LPTV modulators, and cascades of these two. Recall that we deal only with scalar LPTV systems, and that the input and output rates are equal. Thus, Fig. 2, which shows such a system, is completely equivalent to Figs. 15 and 17 with $r = L$, i.e., a uniform maximally decimated filter-bank, as stated in Section III-B.

A. Condition for WSS Outputs from LPTV(L) Systems

We begin by characterizing the bispectra of the outputs of LPTV(L) systems for WSS inputs. For the system of Fig. 2, Appendix D computes the expression for the bispectrum $S_y(e^{j\omega'}, e^{j\omega})$ of the output $y(n)$ in terms of the psd $S_x(e^{j\omega})$ of the WSS input $x(n)$. The result shows that the output is CWSS(L), with an impulsive bispectrum as in Fig. 3(a) and given by

$$\begin{aligned} S_y(e^{j\omega'}, e^{j\omega}) &= \sum_{i=0}^{L-1} P_y^i(e^{j\omega'}) \\ &\quad \sum_{m=-\infty}^{\infty} \delta\left(\omega - \omega' + \frac{2\pi i}{L} + 2\pi m\right), \text{ where} \end{aligned} \tag{37}$$

$$\begin{aligned} P_y^q(e^{j\omega}) = \sum_{r=0}^{L-1} & F_{r+q}(e^{j(\omega+(2\pi q/L))}) \\ & \cdot F_r^*(e^{j\omega}) S_x(e^{j(\omega-(2\pi r/L))}) \end{aligned} \tag{38}$$

Here, the function $P_y^q(e^{j\omega}) = P_y^{q+L}(e^{j\omega})$ describes the shape of the impulse along the qth line $\omega' - \omega = 2\pi q/L$. The $F_q(e^{j\omega})$ are the functions on the impulse-lines of the bifrequency map of the LPTV system.

From the form of the bispectrum of a WSS process [which is shown by Fig. 3(b) and (23)], we know that $y(n)$ is WSS for all WSS $x(n)$ if and only if the following condition is satisfied: $P_y^q(e^{j\omega})$ must be identically zero when q is not a multiple of L for all valid input psd, i.e., for all $S_x(e^{j\omega}) \geq 0$. This is clearly equivalent to

$$\begin{aligned} & F_{r+q}(e^{j(\omega+(2\pi q/L))}) F_r^*(e^{j\omega}) = 0 \\ & \text{for all } r = 0, 1, \cdots, L-1 \end{aligned} \tag{39}$$

whenever q is not a multiple of L. Now, let $A_i(z)$ denote the "aliasing gains" of the LPTV system viewed as a filterbank, as described in Section III-B and given by (8). Then , using $F_q(e^{j\omega}) = A_q(e^{j\omega})$ [see (11)] and reindexing, we get the desired condition

$$\begin{aligned} & A_i(e^{j(\omega+(2\pi(i-r)/L))}) A_r^*(e^{j\omega}) = 0 \\ & \text{for all } i, r \in \{0, 1, \cdots, L-1\} \\ & \text{such that } i \neq r. \end{aligned} \tag{40}$$

1) Conditions from [1] Are Less Explicit: We have shown that a general LPTV(L) scalar system shown in Fig. 2 produces WSS output for all WSS inputs if and only if it satisfies (40), where $A_q(e^{j\omega})$ are the aliasing gains of the system. This condition is not easy to state concisely in an elegant form without using an equation. However, it provides a clear way to test if a given LPTV system has this property or not. Further, it simplifies elegantly in the case of *rational* LPTV systems, as shown in Section VI-B. To contrast this with results from the approach of [1], let $\mathbf{H}(e^{j\omega})$ be the MIMO LTI transfer matrix of the L-fold blocked version of the LPTV system (see

Section II). In addition, let $\mathbf{S_x}(e^{j\omega})$ be the psd matrix of the blocked version $\mathbf{x}^{(L)}(n)$ of $x(n)$ [which is pseudocirculant as $x(n)$ is WSS]. The approach of [1] would use (18) to state the condition as follows: $\mathbf{H}(e^{j\omega})\mathbf{S_x}(e^{j\omega})\mathbf{H}^{\dagger}(e^{j\omega})$ must be pseudocirculant for every pseudocirculant positive semidefinite matrix $\mathbf{S_x}(e^{j\omega})$ [i.e., for every possible valid psd matrix $\mathbf{S_x}(e^{j\omega})$ of $\mathbf{x}^{(L)}(n)$ under the constraint that $x(n)$ is WSS]. This statement gives a very *implicit* condition on the LPTV system and cannot be easily tested.

2) LTI Case: The LPTV(L) system of Fig. 2 becomes LTI with transfer function $B(z)$ if and only if all the filters $B_i(z)$ equal $B(z)$. From (8), this is equivalent to $A_i(z) = 0,\ i = 1, 2, \cdots, L-1$, which means that (40) is satisfied. Therefore, such a system indeed produces WSS output for all WSS inputs, which is a well known fact.

3) Exponential Modulator: A general scalar modulator is a system with output $y(n) = f(n)x(n)$ for input $x(n)$. An exponential modulator is one with $f(n) = e^{-j\omega_0 n}$. This is LPTV(L) if and only if $\omega_0 = 2\pi K/L$ for some integer K. Such a system is shown in Fig. 16(a). We know [1] that any exponential modulator produces a WSS output $y(n)$ for all WSS inputs $x(n)$. To reconcile this result with (40), note that an LPTV(L) exponential modulator can be represented as in Fig. 16(b). This is like the general structure of Fig. 2 with $B_r(z)$ a constant multiplier of value $e^{j2\pi rK/L}$. Therefore, (8) shows that $A_q(z)$ is nonzero (and constant) for exactly one value of $q \in \{0, 1, \cdots, L-1\}$. This means that (40) is indeed satisfied here.

B. Case of Rational LPTV Systems

Rational LPTV(L) systems are systems as in Fig. 2 with all the filters $B_i(z)$ being rational LTI filters. Special cases are rational LTI systems $B(z)$ [where all $B_i(z) = B(z)$] and exponential LPTV(L) modulators $(B_i(z) = e^{j2\pi iK/L})$. Cascades of rational LPTV(L) systems are also rational LPTV(L)—this is evident when we consider the L-fold blocked versions of the systems (which are LTI as seen in Section II).

It is well known that the special cases of rational LTI systems and exponential LPTV(L) modulators produce WSS outputs for all WSS inputs, and hence, so do cascades of these systems. The question arises whether there are other rational LPTV(L) systems with this property. This can be answered using the general condition (40) derived above. We have the following theorem.

Theorem 2: A rational LPTV(L) scalar system produces WSS outputs for all WSS inputs if and only if it is either a rational LTI system, an exponential LPTV(L) modulator, or a cascade of these.

Proof: From the earlier discussion, we see that it suffices to prove the "only if" part of the theorem. We need the relation between the aliasing gains $A_q(e^{j\omega})$ and the filters $B_q(e^{j\omega})$ of the LPTV(L) system. This relation is given by (8) and is reproduced here for convenience:

$$[A_0(z) \quad A_1(z) \quad \cdots \quad A_{L-1}(z)]^T = \frac{1}{L}\,\mathbf{W}_L^{\dagger}[B_0(z)\,B_1(z)\cdots B_{L-1}(z)]^T. \qquad (41)$$

This equation shows that rationality of the $B_i(z)$ is equivalent to that of the $A_i(z)$. Now, consider a rational LPTV(L) system

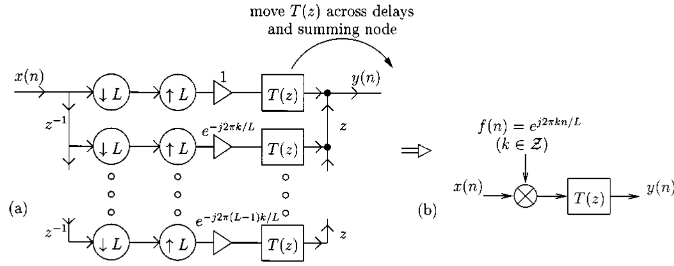

Fig. 18. Illustrating the proof of Theorem 2. (a) Rational LPTV(L) system producing WSS output for all WSS inputs. (b) Equivalent structure for this system.

producing WSS output for all WSS inputs. Thus, $A_i(z)$ are rational and satisfy the condition (40). This is possible if and only if there is at most one $i \in \{0, 1, \cdots, L-1\}$ such that $A_i(z)$ is not identically zero. Excluding the trivial case when there is no such i, let $A_k(z) \triangleq T(z)$ be not identically zero. [$T(z)$ is rational]. Using (41), this means that each $B_i(z)$ in Fig. 2 is the cascade of a constant multiplier $e^{-j2\pi ik/L}$ and $T(z)$. Thus, the system has the structure shown in Fig. 18(a), which shows itself to be equivalent to Fig. 18(b) on comparison with Fig. 16. Thus, the system is indeed a cascade of an exponential LPTV(L) modulator followed by the rational LTI system $T(z)$. This concludes the proof. $\bigtriangledown\bigtriangledown\bigtriangledown$

VII. Other Applications

A. Partial Reconstructions from Subbands of a FB

Principal component filterbanks were proposed in [19], with the idea of compressing the main signal features into a few subbands of the filterbank and dropping the remaining subbands. This results in a partial reconstruction of the signal, and the system producing this reconstruction is equivalent to an overdecimated uniform filterbank, i.e., a system as in Fig. 15, where $r < L$. Theorem 2 then says that given any rational uniform maximally decimated filterbank, none of the partial (or principal component) reconstructions can be WSS for all WSS inputs. For if this were the case, the system creating the reconstruction (an overdecimated rational filterbank) would have to be a cascade of a modulator and an LTI system (from Theorem 2). However, such a cascade is necessarily a *maximally decimated* filterbank, which is a contradiction.

B. Nonrational LPTV Systems

If we relax the restriction of rationality, we can find more examples of LPTV(L) systems producing WSS output for all WSS inputs. The ideal uniform brickwall subband coder is an example of a nonrational filterbank for which the systems producing the partial reconstructions are also LTI. Thus, in this case, every partial reconstruction is WSS if the input is WSS. Systems as in Fig. 2 where the vector of filters $B_q(e^{j\omega})$ is MIMO alias-free (L) are another class of systems that produce WSS output for all WSS inputs, as is clear from using Theorem 1a. Indeed, (41) shows that for such systems the vector of aliasing gains $A_q(e^{j\omega})$ is also MIMO alias-free (L); hence, the condition (40) is satisfied. In fact Theorem 1a shows that this class of systems is also the class for which the output $y(n)$ is WSS for *arbitrary CWSS*(L) scalar inputs $x(n)$.

C. Characterization of CWSS Scalar Bispectra

The equations (37) and (38) give a full characterization of the bispectrum of an arbitrary CWSS(L) scalar process $y(n)$. This is because by spectral factorization of the psd matrix of the blocked version of $y(n)$, we can show that every CWSS(L) scalar process can be obtained by passing (WSS) white noise through an appropriate LPTV(L) system. In particular, this means that $S_y(e^{j\omega'}, e^{j\omega})$ from (37) and (38) will automatically satisfy the properties (15)–(17) for arbitrary transfer functions $F_q(e^{j\omega})$, for any non-negative function (i.e., valid psd) $S_x(e^{j\omega})$.

D. Bifrequency Characterization of Lossless LTV Systems

Lossless LTI systems have been extensively studied in the literature, owing to connections with paraunitary filterbanks and orthonormal wavelets. Losslessness of a causal stable LTI system $\mathbf{H}(z)$ may be described by two equivalent definitions.

1) The system input and output "energies" (l^2 norms) are always equal.
2) The inverse of the system is $\tilde{\mathbf{H}}(z)$, which is the so-called paraconjugate, defined so that $\tilde{\mathbf{H}}(e^{j\omega}) = \mathbf{H}^\dagger(e^{j\omega})$.

The extension to LTV systems is described in [5] and [6] in connection with time-varying paraunitary filterbanks. Using the notation of [5], the above definitions apply and are shown to be equivalent for LTV systems as well, except that we have a new definition of the "paraconjugate" $\tilde{\mathbf{E}}(n, \mathcal{Z})$ of the LTV system $\mathbf{E}(n, \mathcal{Z})$. Converting this definition from [5] into the notation using the Green's function $\mathbf{k}(m, n)$, we find that the paraconjugate is the system with Green's function $\mathbf{g}(m, n) = \mathbf{k}^\dagger(n, m)$. This system is called the adjoint or dual system in [2]. Thus, using (5) and (6), the second definition above for a lossless LTV system $\mathbf{k}(m, n)$ now reads as

$$\sum_{r=-\infty}^{\infty} \mathbf{k}^\dagger(r, m)\mathbf{k}(r, n) = \mathbf{I}\delta_0(m - n) \tag{42}$$

$$\int_{-\pi}^{\pi} \mathbf{K}^\dagger(e^{j\omega''}, e^{j\omega'})\, \mathbf{K}(e^{j\omega''}, e^{j\omega})\, d\omega'' = \mathbf{I} \sum_{q=-\infty}^{\infty} \delta(\omega - \omega' + 2\pi q). \tag{43}$$

(Here, $\delta_0(n)$ is the discrete impulse sequence.) This notation is more general since the notation of [5] is useful only if the input and output rates of the LTV system are equal. For example, the L-fold expander is a lossless LTV system that necessarily requires this notation. For this system, $\mathbf{k}(m, n) = \delta_0(m - Ln)$ [2], which is easily seen to satisfy (42). This is consistent with the fact that the expander is lossless.

The term "adjoint" here has the same meaning as in operator theory: If V_1, V_2 are inner-product spaces, the adjoint of a linear operator A: $V_1 \rightarrow V_2$ is the operator A_*: $V_2 \rightarrow V_1$ satisfying $\langle Ax, y\rangle = \langle x, A_* y\rangle$ for all $x \in V_1, y \in V_2$ (where $\langle v_1, v_2\rangle$ is the innerproduct of the two vectors v_1, v_2 from the same inner-product space). Now, viewing the LTV system $\mathbf{k}(m, n)$ as an operator from $l^2(M)$ to $l^2(N)$, we can show on lines similar to [5] that the adjoint of this operator is the LTV system $\mathbf{k}^\dagger(n, m)$. Thus, general lossless LTV systems are *operators whose inverses are their adjoints*.

VIII. Concluding Remarks

We have used bifrequencies and bispectra to study the effects of multirate systems on the statistics of random input signals. We have shown that this often yields more insight into the working of these systems than other approaches based on polyphase matrices and pseudocirculants. It allows easy generalization of many of the results of [1] to MIMO systems. It allows us to prove two nontrivial results (Theorems 1 and 2) and seems to be a powerful tool for answering questions like "When does a multirate system produce WSS outputs for all WSS inputs?" However, as with any tool, indiscriminate use of bifrequency analysis may be inefficient in many situations. What appears to make it especially useful is the geometric insight obtained when LPTV systems and CWSS processes are involved, causing the 2-D Fourier transforms to become impulsive lines.

One question arising from our analysis is the following: Can these results be used to tell us more about lossless LTV systems described in [5]? We have pointed out the bifrequency characterization of a general lossless LTV system. However, the bifrequency function is so general that it is not immediately clear whether this helps in any way. A special case that might be considered is that of lossless LPTV systems. Scalar systems of this kind have been dealt with in this paper. Vector systems correspond to paraunitary periodically time-varying filterbanks, where the analysis and synthesis filterbanks are switched cyclically between a selection of banks. Bifrequency analysis of these may lead to some new insights. Another special case might be that of "rational" LTV systems, i.e., those realizable by a linear difference equation with time-varying coefficients. The bifrequency function for such a system, which is given by (4), clearly reflects the fact that it is fully characterized by the coefficients of the difference equation (unlike the representation in [5]). Another area where bifrequency analysis might be useful is in the theory of matrix filterbanks described in [20]. We could also ask other questions further generalizing the issues considered here, e.g., find all LTV (as opposed to LPTV) systems producing WSS outputs for all WSS inputs. The answer to this is not directly evident from using bispectra. For example, arbitrary exponential modulators (not necessarily LPTV) also fall in this class [1].

Appendix A

Here, we prove the rule expressed by Fig. 7 for determining the effect of a general LTV system on the bispectrum of its input. Let the processes $\mathbf{x}(n)$ and $\mathbf{y}(n)$ be, respectively, the input and output of an LTV system represented by its Green's function $\mathbf{k}(m, n)$. Equations (1) and (2) are therefore satisfied. Define the autocorrelation sequences $\mathbf{r_x}(m,n) = E[\mathbf{x}(m)\mathbf{x}^\dagger(n)]$ and $\mathbf{r_y}(m,n) = E[\mathbf{y}(m)\mathbf{y}^\dagger(n)]$. Using (1), we have

$$\begin{aligned}\mathbf{r_y}(m,n) &= E\left[\sum_{p=-\infty}^{\infty}\mathbf{k}(m,p)\mathbf{x}(p)\left(\sum_{q=-\infty}^{\infty}\mathbf{k}(n,q)\mathbf{x}(q)\right)^\dagger\right]\\ &= \sum_{p=-\infty}^{\infty}\mathbf{k}(m,p)\sum_{q=-\infty}^{\infty}E[\mathbf{x}(p)\mathbf{x}^\dagger(q)]\,\mathbf{k}^\dagger(n,q)\\ &= \sum_{p=-\infty}^{\infty}\mathbf{k}(m,p)\mathbf{f}(p,n), \quad \text{where} \qquad (44)\\ \mathbf{f}(p,n) &= \sum_{q=-\infty}^{\infty}\mathbf{r_x}(p,q)\mathbf{k}^\dagger(n,q). \qquad (45)\end{aligned}$$

By comparison with (5), $\mathbf{f}(p,n)$ in (45) is the Green's function of the cascade of the LTV systems with Green's functions $\mathbf{g}(m,n) \triangleq \mathbf{k}^\dagger(n,m)$ and $\mathbf{r_x}(m,n)$ (in that order). Similarly, the output autocorrelation $\mathbf{r_y}(m,n)$ in (44) is the Green's function of the cascade of the LTV systems represented by $\mathbf{f}(m,n)$ and $\mathbf{k}(m,n)$ (in that order). Hence, the 2-D Fourier transform of $\mathbf{r_y}(m,n)$, i.e., the output bispectrum, is the bifrequency function of this cascade. Drawing the cascades and replacing Green's functions with bifrequency functions yields Fig. 7.

APPENDIX B

This appendix serves to show that if the filterbank in Fig. 15 has the PR property, the vector $\mathbf{x}(n)$ input to its synthesis section is *not* an arbitrary WSS vector process even if the filterbank input $w(n)$ is an arbitrary WSS scalar process. To do this, we redraw Fig. 15 as in Fig. 17, where $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are the analysis and synthesis polyphase matrices of the filterbank. We can now use (18) to show that the psd matrix of $\mathbf{x}(n)$ is $\mathbf{S_x}(e^{j\omega}) = \mathbf{E}(e^{j\omega})\mathbf{S_w}(e^{j\omega})\mathbf{E}^\dagger(e^{j\omega})$. Here, $\mathbf{S_w}(e^{j\omega})$ is the psd matrix of the blocked vector $\mathbf{w}^{(L)}(n)$ in Fig. 17, and is thus pseudocirculant since $w(n)$ is WSS. Now, PR implies that $\mathbf{E}(e^{j\omega})$ is invertible (for all ω); hence, $\mathbf{S_x}(e^{j\omega})$ cannot be an arbitrary positive semidefinite matrix. For example, if $\mathbf{A}(e^{j\omega})$ is positive semidefinite but not pseudocirculant, then $\mathbf{E}(e^{j\omega})\mathbf{A}(e^{j\omega})\mathbf{E}^\dagger(e^{j\omega})$ is an example of a positive semidefinite matrix that $\mathbf{S_x}(e^{j\omega})$ cannot equal because $\mathbf{S_w}(e^{j\omega})$ cannot equal $\mathbf{A}(e^{j\omega})$.

APPENDIX C

Lemma 1: If $\mathbf{P},\mathbf{Q}$ are fixed matrices of the same size such that $\mathbf{P}^\dagger\mathbf{A}\mathbf{Q} = \mathbf{0}$ for every Hermitian positive definite matrix $\mathbf{A}$, then either $\mathbf{P} = 0$ or $\mathbf{Q} = 0$ (or both).

To prove the lemma, first consider the case when $\mathbf{P} = \mathbf{w}$ and $\mathbf{Q} = \mathbf{v}$ are column vectors. Assuming both are nonzero, we will establish a contradiction. Choosing $\mathbf{A} = \mathbf{I}$ shows that the vectors $\mathbf{v},\mathbf{w}$ are orthogonal, i.e., $\mathbf{w}^\dagger\mathbf{v} = 0$. However, we can easily find a linear transform that acts on these vectors to produce two nonorthogonal nonzero vectors. To be specific, let $\mathbf{x},\mathbf{y}$ be any two independent nonzero nonorthogonal column vectors of same size as $\mathbf{v},\mathbf{w}$. Then, we can find a nonsingular square matrix $\mathbf{B}$ such that $\mathbf{Bv} = \mathbf{x}$ and $\mathbf{Bw} = \mathbf{y}$. Since $\mathbf{y}^\dagger\mathbf{x} \neq 0$, taking $\mathbf{A} = \mathbf{B}^\dagger\mathbf{B}$ (which is Hermitian positive definite) yields $\mathbf{w}^\dagger\mathbf{A}\mathbf{v} \neq 0$, which is a contradiction. [To find the matrix $\mathbf{B}$ here, suppose $\mathbf{v},\mathbf{w} \in \mathcal{R}^n$, which is the space of all n-tuples. Extend the linearly independent set $\{\mathbf{v},\mathbf{w}\}$ to a basis of $\mathcal{R}^n$, and use the basis elements as columns of a matrix $\mathbf{C}$. Thus, $\mathbf{C}$ is nonsingular with first two columns $\mathbf{v},\mathbf{w}$. Repeat the same construction starting with $\{\mathbf{x},\mathbf{y}\} \subset \mathcal{R}^n$ to get a nonsingular matrix $\mathbf{D}$ with first two columns $\mathbf{x},\mathbf{y}$. Then, $\mathbf{B} = \mathbf{D}\mathbf{C}^{-1}$ is a satisfactory choice.]

Next, we use the above to prove by contradiction the general case, i.e., $\mathbf{P} = [\mathbf{w}_0, \mathbf{w}_1, \cdots, \mathbf{w}_{N-1}]$, and $\mathbf{Q} = [\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_{N-1}]$, where $\mathbf{w}_i, \mathbf{v}_i, i = 0,1,\cdots,N-1$ are column vectors of the same size. If both $\mathbf{P},\mathbf{Q}$ are nonzero, we can find $i,j \in \{0,1,\cdots,N-1\}$ such that both $\mathbf{w}_i$ and $\mathbf{v}_j$ are nonzero. Now, for all Hermitian positive definite $\mathbf{A}$, $\mathbf{P}^\dagger\mathbf{A}\mathbf{Q} = \mathbf{0}$, and therefore, its (i,j) entry $\mathbf{w}_i^\dagger\mathbf{A}\mathbf{v}_j$ is zero as well. This is impossible by the established statement of the lemma when $\mathbf{P},\mathbf{Q}$ are column vectors. This completes the proof.

APPENDIX D

This appendix serves to prove (37) and (38). As explained in Section III-D, $S_y(e^{j\omega'}, e^{j\omega})$ is the bifrequency function of the cascade of Fig. 7, where $\mathbf{K}(e^{j\omega'}, e^{j\omega}) = K(e^{j\omega'}, e^{j\omega})$ is a scalar given by (9), and

$$\begin{aligned}\mathbf{S_x}(e^{j\omega'}, e^{j\omega}) &= S_x(e^{j\omega'}, e^{j\omega})\\ &= S_x(e^{j\omega})\sum_{m=-\infty}^{\infty}\delta(\omega - \omega' + 2\pi m) \qquad (46)\end{aligned}$$

which is (23) in scalar form. By (6), the cascade of $K^\dagger(e^{j\omega}, e^{j\omega'})$ and $S_x(e^{j\omega'}, e^{j\omega})$ has bifrequency function

$$\begin{aligned}G(e^{j\omega'}, e^{j\omega}) &= \int_{-\pi}^{\pi} S_x(e^{j\omega'}, e^{j\omega''})K^\dagger(e^{j\omega}, e^{j\omega''})\,d\omega'' \qquad (47)\\ &= \int_{-\pi}^{\pi} S_x(e^{j\omega'}, e^{j\omega''})\,F^*(e^{j\omega}, e^{j\omega''})\\ &\quad \cdot \sum_{r=-\infty}^{\infty}\delta\left(\omega'' - \omega + \frac{2\pi r}{L}\right)d\omega''. \qquad (48)\end{aligned}$$

Using the sifting property of the $\delta(\cdot)$ function and the definition $F_r(e^{j\omega}) = F(e^{j\omega}, e^{j(\omega-(2\pi r/L))})$, we get

$$\begin{aligned}G(e^{j\omega'}, e^{j\omega}) &= \sum_{r=0}^{L-1} S_x(e^{j\omega'}, e^{j(\omega-(2\pi r/L))})\\ &\quad \cdot F^*(e^{j\omega}, e^{j(\omega-(2\pi r/L))}) \qquad (49)\\ &= \sum_{r=0}^{L-1} S_x(e^{j\omega'}, e^{j(\omega-(2\pi r/L))})\,F_r^*(e^{j\omega}). \qquad (50)\end{aligned}$$

One more application of (6) to cascade $G(e^{j\omega'}, e^{j\omega})$ with $K(e^{j\omega'}, e^{j\omega})$ yields

$$\begin{aligned}S_y(e^{j\omega'}, e^{j\omega}) &= \int_{-\pi}^{\pi} F(e^{j\omega'}, e^{j\omega''})\\ &\quad \cdot \left(\sum_{i=-\infty}^{\infty}\delta\left(\omega'' - \omega' + \frac{2\pi i}{L}\right)\right)\\ &\quad \cdot G(e^{j\omega''}, e^{j\omega})\,d\omega'' \qquad (51)\\ &= \sum_{i=0}^{L-1} F(e^{j\omega'}, e^{j(\omega'-(2\pi i/L))})G(e^{j(\omega'-(2\pi i/L))}, e^{j\omega}). \qquad (52)\end{aligned}$$

Using (46) and (50), we get

$$S_y(e^{j\omega'}, e^{j\omega}) = \sum_{i=0}^{L-1} \sum_{r=0}^{L-1} F_i(e^{j\omega'}) \cdot S_x(e^{j(\omega'-(2\pi i/L))}, e^{j(\omega-(2\pi r/L))}) F_r^*(e^{j\omega}) \quad (53)$$

$$= \sum_{i=0}^{L-1} \sum_{r=0}^{L-1} F_i(e^{j\omega'}) F_r^*(e^{j\omega}) S_x(e^{j(\omega-(2\pi r/L))}) \cdot \sum_{m=-\infty}^{\infty} \delta\left(\omega - \omega' + \frac{2\pi(i-r)}{L} + 2\pi m\right). \quad (54)$$

We now use the L-fold periodicity of the summand in the index i to replace i with $i+r$ throughout the summand. The inner sum over m then becomes independent of r. Using $f(x)\delta(x) = f(0)\delta(x)$, this yields

$$S_y(e^{j\omega'}, e^{j\omega}) = \sum_{i=0}^{L-1} \left(\sum_{r=0}^{L-1} F_{i+r}(e^{j(\omega+(2\pi i/L))}) F_r^*(e^{j\omega}) S_x(e^{j(\omega-(2\pi r/L))}) \right) \cdot \sum_{m=-\infty}^{\infty} \delta\left(\omega - \omega' + \frac{2\pi i}{L} + 2\pi m\right) \quad (55)$$

This is identical to (37) with $P_y^q(e^{j\omega})$ defined as in (38). This completes the derivation of (37) and (38).

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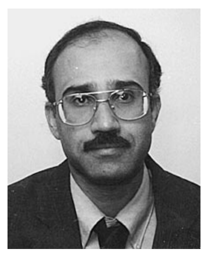

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