FNSNF
Fonds national suisse
Schweizerischer Nationalfonds
Fondo nazionale svizzero
Swiss National Science Foundation

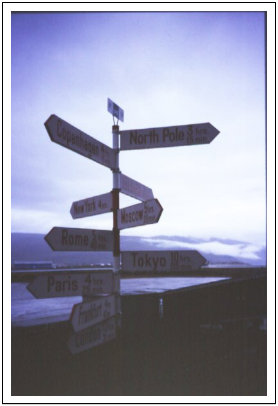
Algebraic Cayley differential space-time codes

Frédérique Oggier
(joint work with Babak Hassibi)
frederique@systems.caltech.edu

California Institute of Technology

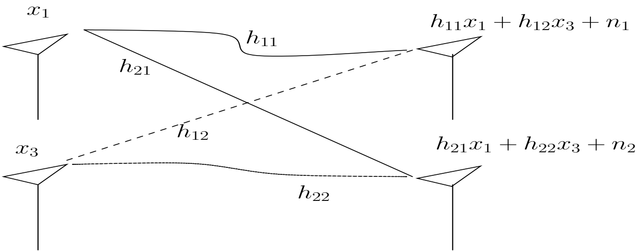
Sequences and Codes, Vancouver, July 17th 2006

Outline

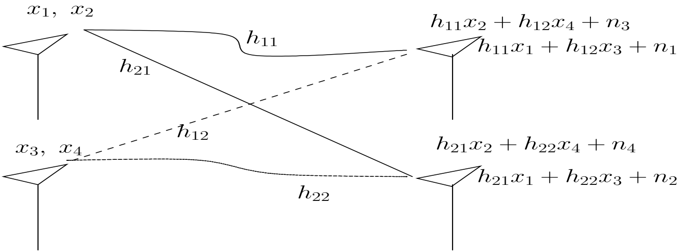


Space-Time Coding

Space-Time Coding

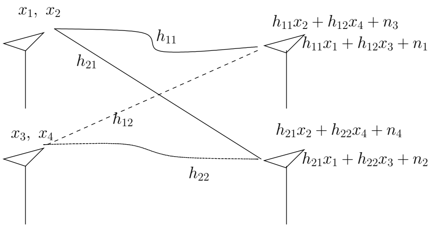


Space-Time Coding



Space-Time Coding: the model

$$\mathbf{Y} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} + \mathbf{W},\ \mathbf{W},\ \mathbf{H} \text{ complex Gaussian}$$



The code design

The goal is the design of the codebook $\mathcal{C}$:

$$\mathcal{C} = \left\{ \mathbf{X} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} | x_1, x_2, x_3, x_4 \in \mathbb{C} \right\}$$

the x_i are functions of the information symbols.

- The *pairwise probability of error* of sending $\mathbf{X}$ and decoding $\hat{\mathbf{X}} \neq \mathbf{X}$ is upper bounded by

$$P(\mathbf{X} \rightarrow \hat{\mathbf{X}}) \leq \ ?$$

The code design

The goal is the design of the codebook $\mathcal{C}$:

$$\mathcal{C} = \left\{ \mathbf{X} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} | x_1, x_2, x_3, x_4 \in \mathbb{C} \right\}$$

the x_i are functions of the information symbols.

- The *pairwise probability of error* of sending $\mathbf{X}$ and decoding $\hat{\mathbf{X}} \neq \mathbf{X}$ is upper bounded by

$$P(\mathbf{X} \rightarrow \hat{\mathbf{X}}) \leq \ ?$$

Coherent vs noncoherent MIMO channel

- Let us assume the receiver knows the channel (which is called *coherent* case). Then we have

$$P(\mathbf{X} \to \hat{\mathbf{X}}) = P(\|\mathbf{H}\mathbf{X} - \mathbf{Y}\| \geq \|\mathbf{H}\hat{\mathbf{X}} - \mathbf{Y}\|)$$

- Assume now the receiver does *not* know the channel (which is called *noncoherent* case).
- How to do decoding?

Coherent vs noncoherent MIMO channel

- Let us assume the receiver knows the channel (which is called *coherent* case). Then we have

$$P(\mathbf{X} \to \hat{\mathbf{X}}) = P(\|\mathbf{HX} - \mathbf{Y}\| \geq \|\mathbf{H}\hat{\mathbf{X}} - \mathbf{Y}\|)$$

- Assume now the receiver does *not* know the channel (which is called *noncoherent* case).
- How to do decoding?

The differential noncoherent MIMO channel

- We use *differential unitary space-time modulation*. that is (assuming $\mathbf{S}_0 = \mathbf{I}$)

$$\mathbf{S}_t = \mathbf{X}_{z_t}\mathbf{S}_{t-1},\ \ t = 1, 2, \dots,$$

  where $z_t \in \{0, \dots, L-1\}$ is the data to be transmitted, and $\mathcal{C} = \{\mathbf{X}_0, \dots, \mathbf{X}_{L-1}\}$ the constellation to be designed.
- The matrices $\mathbf{X}$ have to be *unitary*.

The differential noncoherent MIMO channel

- We use *differential unitary space-time modulation*. that is (assuming $\mathbf{S}_0 = \mathbf{I}$)

$$\mathbf{S}_t = \mathbf{X}_{z_t}\mathbf{S}_{t-1},\ t = 1, 2, \ldots,$$

  where $z_t \in \{0, \ldots, L-1\}$ is the data to be transmitted, and $\mathcal{C} = \{\mathbf{X}_0, \ldots, \mathbf{X}_{L-1}\}$ the constellation to be designed.
- The matrices $\mathbf{X}$ have to be *unitary*.

The decoding

- If we assume the channel is roughly constant, we have

$$\begin{aligned} \mathbf{Y}_t &= \mathbf{S}_t\mathbf{H} + \mathbf{W}_t \\ &= \mathbf{X}_{z_t}\mathbf{S}_{t-1}\mathbf{H} + \mathbf{W}_t \\ &= \mathbf{X}_{z_t}(\mathbf{Y}_{t-1} - \mathbf{W}_{t-1}) + \mathbf{W}_t \\ &= \mathbf{X}_{z_t}\mathbf{Y}_{t-1} + \mathbf{W}'_t. \end{aligned}$$

- The matrix $\mathbf{H}$ does *not* appear in the last equation.
- The decoder is thus given by

$$\hat{z}_t = \arg\min_{l=0,\ldots,|\mathcal{C}|-1} \|\mathbf{Y}_t - \mathbf{X}_l\mathbf{Y}_{t-1}\|.$$

The decoding

- If we assume the channel is roughly constant, we have

$$\begin{aligned}\mathbf{Y}_t &= \mathbf{S}_t\mathbf{H} + \mathbf{W}_t \\ &= \mathbf{X}_{z_t}\mathbf{S}_{t-1}\mathbf{H} + \mathbf{W}_t \\ &= \mathbf{X}_{z_t}(\mathbf{Y}_{t-1} - \mathbf{W}_{t-1}) + \mathbf{W}_t \\ &= \mathbf{X}_{z_t}\mathbf{Y}_{t-1} + \mathbf{W}'_t.\end{aligned}$$

- The matrix **H** does *not* appear in the last equation.
- The decoder is thus given by

$$\hat{z}_t = \arg \min_{l=0,\ldots,|\mathcal{C}|-1} \|\mathbf{Y}_t - \mathbf{X}_l\mathbf{Y}_{t-1}\|.$$

The decoding

- If we assume the channel is roughly constant, we have

$$\begin{aligned}\mathbf{Y}_t &= \mathbf{S}_t\mathbf{H} + \mathbf{W}_t \\ &= \mathbf{X}_{z_t}\mathbf{S}_{t-1}\mathbf{H} + \mathbf{W}_t \\ &= \mathbf{X}_{z_t}(\mathbf{Y}_{t-1} - \mathbf{W}_{t-1}) + \mathbf{W}_t \\ &= \mathbf{X}_{z_t}\mathbf{Y}_{t-1} + \mathbf{W}'_t.\end{aligned}$$

- The matrix **H** does *not* appear in the last equation.
- The decoder is thus given by

$$\hat{z}_t = \arg\min_{l=0,\ldots,|\mathcal{C}|-1} \|\mathbf{Y}_t - \mathbf{X}_l\mathbf{Y}_{t-1}\|.$$

Probability of error

- At high SNR, the *pairwise probability of error* P_e has the upper bound

$$P_e \leq \left(\frac{1}{2}\right)\left(\frac{8}{\rho}\right)^{MN}\frac{1}{|\det(\mathbf{X}_i-\mathbf{X}_j)|^{2N}}$$

- The quality of the code is measure by the *diversity product*

$$\zeta_{\mathcal{C}} = \frac{1}{2}\min_{\mathbf{X}_i\neq\mathbf{X}_j}|\det(\mathbf{X}_i-\mathbf{X}_j)|^{1/M} \qquad \forall\, \mathbf{X}_i \neq \mathbf{X}_j \in \mathcal{C}$$

Problem statement

- Find a set $\mathcal{C}$ of *unitary* matrices ($\mathbf{X}\mathbf{X}^\dagger = \mathbf{I}$) such that

$$\det(\mathbf{X}_i - \mathbf{X}_j) \neq 0 \qquad \forall\ \mathbf{X}_i \neq \mathbf{X}_j \in \mathcal{C}$$

- Find a way of *encoding* and *decoding* these matrices.

Problem statement

- Find a set $\mathcal{C}$ of *unitary* matrices ($\mathbf{X}\mathbf{X}^\dagger = \mathbf{I}$) such that

$$\det(\mathbf{X}_i - \mathbf{X}_j) \neq 0 \qquad \forall\ \mathbf{X}_i \neq \mathbf{X}_j \in \mathcal{C}$$

- Find a way of *encoding* and *decoding* these matrices.

The Cayley transform

- Let A be an Hermitian matrix, that is $A^{\dagger} = A$.
- Its *Cayley transform* is given by

$$V = (\mathbf{I} + iA)^{-1}(\mathbf{I} - iA).$$

Encoding a Cayley code

- Let $\alpha_1, \dots, \alpha_Q \in \mathcal{S} \subset \mathbb{R}$ be the information symbols.
- Let $A_1, \dots, A_Q$ be a basis of Q Hermitian matrices.
- Encode the α_i's in A:

$$A = \sum_{q=1}^{Q} \alpha_q A_q.$$

- Compute

$$V = (\mathbf{I} + iA)^{-1}(\mathbf{I} - iA).$$

Encoding a Cayley code

- Let $\alpha_1, \dots, \alpha_Q \in \mathcal{S} \subset \mathbb{R}$ be the information symbols.
- Let $A_1, \dots, A_Q$ be a basis of Q Hermitian matrices.
- Encode the α_i's in A:

$$A = \sum_{q=1}^{Q} \alpha_q A_q.$$

- Compute

$$V = (\mathbf{I} + iA)^{-1}(\mathbf{I} - iA).$$

Design criteria for Cayley codes

- *Recall* we want $\det(V_i - V_j) \neq 0$, which is equivalent to ask

$$\det(A_i - A_j) \neq 0, \quad i \neq j,$$

where A_i are *Hermitian*.

- The rate of the code is

$$\frac{Q}{M} \log |\mathcal{S}|.$$

Design criteria for Cayley codes

- *Recall* we want $\det(V_i - V_j) \neq 0$, which is equivalent to ask

$$\det(A_i - A_j) \neq 0, \quad i \neq j,$$

where A_i are *Hermitian*.

- The rate of the code is

$$\frac{Q}{M} \log |\mathcal{S}|.$$

Previous Cayley codes

- Cayley codes were introduced by Hassibi and Hochwald.
- They are available at *high rate*.
- The diversity criterion was replaced by an information theoretical criterion.
- Cayley codes can be efficiently decoded (*linearized Sphere Decoder*).
- One drawback: *heavy optimization* is required for each number of antennas and each rate.

Previous Cayley codes

- Cayley codes were introduced by Hassibi and Hochwald.
- They are available at *high rate*.
- The diversity criterion was replaced by an information theoretical criterion.
- Cayley codes can be efficiently decoded (*linearized Sphere Decoder*).
- One drawback: *heavy optimization* is required for each number of antennas and each rate.

The first ingredient: linearity

- The difficulty in building $\mathcal{C}$ such that

$$\det(\mathbf{X}_i - \mathbf{X}_j) \neq 0, \ \mathbf{X}_i \neq \mathbf{X}_j \in \mathcal{C},$$

comes from the *non-linearity* of the determinant.

- An algebra of matrices is *linear*, so that

$$\det(\mathbf{X}_i - \mathbf{X}_j) = \det(\mathbf{X}_k),$$

$\mathbf{X}_k$ a matrix in the algebra.

The first ingredient: linearity

- The difficulty in building $\mathcal{C}$ such that

$$\det(\mathbf{X}_i - \mathbf{X}_j) \neq 0, \; \mathbf{X}_i \neq \mathbf{X}_j \in \mathcal{C},$$

comes from the *non-linearity* of the determinant.

- An algebra of matrices is *linear*, so that

$$\det(\mathbf{X}_i - \mathbf{X}_j) = \det(\mathbf{X}_k),$$

$\mathbf{X}_k$ a matrix in the algebra.

The second ingredient: invertibility

- The problem is now to build a family $\mathcal{C}$ of matrices such that

$$\det(\mathbf{X}) \neq 0, \ \mathbf{0} \neq \mathbf{X} \in \mathcal{C}.$$

or equivalently, such that each $\mathbf{0} \neq \mathbf{X} \in \mathcal{C}$ is *invertible*.

- By definition, a *field* is a set such that every (nonzero) element in it is invertible.
- Take $\mathcal{C}$ inside an algebra of matrices which is also a field.
- A *division algebra* is a non-commutative field.

The second ingredient: invertibility

- The problem is now to build a family $\mathcal{C}$ of matrices such that

$$\det(\mathbf{X}) \neq 0, \ \mathbf{0} \neq \mathbf{X} \in \mathcal{C}.$$

  or equivalently, such that each $\mathbf{0} \neq \mathbf{X} \in \mathcal{C}$ is *invertible*.
- By definition, a *field* is a set such that every (nonzero) element in it is invertible.
- Take $\mathcal{C}$ inside an algebra of matrices which is also a field.
- A *division algebra* is a non-commutative field.

The second ingredient: invertibility

- The problem is now to build a family $\mathcal{C}$ of matrices such that

$$\det(\mathbf{X}) \neq 0, \ \mathbf{0} \neq \mathbf{X} \in \mathcal{C}.$$

  or equivalently, such that each $\mathbf{0} \neq \mathbf{X} \in \mathcal{C}$ is *invertible*.
- By definition, a *field* is a set such that every (nonzero) element in it is invertible.
- Take $\mathcal{C}$ inside an algebra of matrices which is also a field.
- A *division algebra* is a non-commutative field.

The second ingredient: invertibility

- The problem is now to build a family $\mathcal{C}$ of matrices such that

$$\det(\mathbf{X}) \neq 0,\ \mathbf{0} \neq \mathbf{X} \in \mathcal{C}.$$

  or equivalently, such that each $\mathbf{0} \neq \mathbf{X} \in \mathcal{C}$ is *invertible*.
- By definition, a *field* is a set such that every (nonzero) element in it is invertible.
- Take $\mathcal{C}$ inside an algebra of matrices which is also a field.
- A *division algebra* is a non-commutative field.

An example of division algebras: cyclic division algebras

- Let $\mathbb{Q}(i) = \{a + ib,\ a, b\ \in \mathbb{Q}\}$.
- Let L be a vector space of dimension n over $\mathbb{Q}(i)$.
- A *cyclic algebra* $\mathcal{A}$ is defined as follows

$$\mathcal{A} = \{(x_0, x_1, \ldots, x_{n-1}) \mid x_i \in L\}$$

  with basis $\{1, e, \ldots, e^{n-1}\}$ and $e^n = \gamma \in \mathbb{Q}(i)$.
- Think of $i^2 = -1$.

An example of division algebras: cyclic division algebras

- Let $\mathbb{Q}(i) = \{a + ib,\ a, b\ \in \mathbb{Q}\}$.
- Let L be a vector space of dimension n over $\mathbb{Q}(i)$.
- A *cyclic algebra* $\mathcal{A}$ is defined as follows

$$\mathcal{A} = \{(x_0, x_1, \ldots, x_{n-1}) \mid x_i \in L\}$$

with basis $\{1, e, \ldots, e^{n-1}\}$ and $e^n = \gamma \in \mathbb{Q}(i)$.

- Think of $i^2 = -1$.

An example of division algebras: cyclic division algebras

- Let $\mathbb{Q}(i) = \{a + ib,\ a, b\ \in \mathbb{Q}\}$.
- Let L be a vector space of dimension n over $\mathbb{Q}(i)$.
- A *cyclic algebra* $\mathcal{A}$ is defined as follows

$$\mathcal{A} = \{(x_0, x_1, \ldots, x_{n-1}) \mid x_i \in L\}$$

with basis $\{1, e, \ldots, e^{n-1}\}$ and $e^n = \gamma \in \mathbb{Q}(i)$.

- Think of $i^2 = -1$.

An example of division algebras: cyclic division algebras

- Let $\mathbb{Q}(i) = \{a + ib,\ a, b\ \in \mathbb{Q}\}$.
- Let L be a vector space of dimension n over $\mathbb{Q}(i)$.
- A *cyclic algebra* $\mathcal{A}$ is defined as follows

$$\mathcal{A} = \{(x_0, x_1, \ldots, x_{n-1}) \mid x_i \in L\}$$

  with basis $\{1, e, \ldots, e^{n-1}\}$ and $e^n = \gamma \in \mathbb{Q}(i)$.
- Think of $i^2 = -1$.

Cyclic algebras: how to multiply

1. For $n = 2$, $x \in \mathcal{A}$ can be written $x = x_0 + ex_1$.
2. Compute the multiplication by x of any element $y \in \mathcal{A}$.

$$\begin{aligned} xy &= (x_0 + ex_1)(y_0 + ey_1) \\ &= x_0 y_0 + x_0 e y_1 + e x_1 y_0 + e x_1 e y_1 \end{aligned}$$

3. The *noncommutativity rule*: $\lambda e = e\sigma(\lambda)$, $\sigma : L \to L$ a "suitable" map .
4. So that

$$\begin{aligned} xy &= x_0 y_0 + e\sigma(x_0) y_1 + e x_1 y_0 + \gamma\sigma(x_1) y_1 \\ &= [x_0 y_0 + \gamma\sigma(x_1) y_1] + e[\sigma(x_0) y_1 + x_1 y_0], \end{aligned}$$

since $e^2 = \gamma$.

Cyclic algebras: how to multiply

1. For $n = 2$, $x \in \mathcal{A}$ can be written $x = x_0 + ex_1$.
2. Compute the multiplication by x of any element $y \in \mathcal{A}$.

$$\begin{aligned} xy &= (x_0 + ex_1)(y_0 + ey_1) \\ &= x_0 y_0 + x_0 e y_1 + e x_1 y_0 + e x_1 e y_1 \end{aligned}$$

3. The *noncommutativity rule*: $\lambda e = e\sigma(\lambda)$, $\sigma : L \to L$ a "suitable" map .
4. So that

$$\begin{aligned} xy &= x_0 y_0 + e\sigma(x_0) y_1 + e x_1 y_0 + \gamma\sigma(x_1) y_1 \\ &= [x_0 y_0 + \gamma\sigma(x_1) y_1] + e[\sigma(x_0) y_1 + x_1 y_0], \end{aligned}$$

since $e^2 = \gamma$.

Cyclic algebras: how to multiply

1. For $n = 2$, $x \in \mathcal{A}$ can be written $x = x_0 + ex_1$.
2. Compute the multiplication by x of any element $y \in \mathcal{A}$.

$$\begin{aligned} xy &= (x_0 + ex_1)(y_0 + ey_1) \\ &= x_0y_0 + x_0ey_1 + ex_1y_0 + ex_1ey_1 \end{aligned}$$

3. The *noncommutativity rule*: $\lambda e = e\sigma(\lambda)$, $\sigma : L \to L$ a "suitable" map .
4. So that

$$\begin{aligned} xy &= x_0y_0 + e\sigma(x_0)y_1 + ex_1y_0 + \gamma\sigma(x_1)y_1 \\ &= [x_0y_0 + \gamma\sigma(x_1)y_1] + e[\sigma(x_0)y_1 + x_1y_0], \end{aligned}$$

since $e^2 = \gamma$.

Cyclic algebras: how to multiply

1. For $n = 2$, $x \in \mathcal{A}$ can be written $x = x_0 + ex_1$.
2. Compute the multiplication by x of any element $y \in \mathcal{A}$.

$$
\begin{aligned}
xy &= (x_0 + ex_1)(y_0 + ey_1) \\
&= x_0 y_0 + x_0 e y_1 + e x_1 y_0 + e x_1 e y_1
\end{aligned}
$$

3. The *noncommutativity rule*: $\lambda e = e\sigma(\lambda)$, $\sigma : L \to L$ a "suitable" map .
4. So that

$$
\begin{aligned}
xy &= x_0 y_0 + e\sigma(x_0) y_1 + e x_1 y_0 + \gamma\sigma(x_1) y_1 \\
&= [x_0 y_0 + \gamma\sigma(x_1) y_1] + e[\sigma(x_0) y_1 + x_1 y_0],
\end{aligned}
$$

since $e^2 = \gamma$.

Cyclic algebras: matrix formulation

1. We have $xy = [x_0y_0 + \gamma\sigma(x_1)y_1] + e[\sigma(x_0)y_1 + x_1y_0]$.
2. In the basis $\{1, e\}$, this yields

$$xy = \begin{pmatrix} x_0 & \gamma\sigma(x_1) \\ x_1 & \sigma(x_0) \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}.$$

3. There is thus a correspondance

$$x = x_0 + ex_1 \in \mathcal{A} \leftrightarrow \begin{pmatrix} x_0 & \gamma\sigma(x_1) \\ x_1 & \sigma(x_0) \end{pmatrix}.$$

4. We associate to an element its *multiplication matrix*.

Cyclic algebras: matrix formulation

1. We have $xy = [x_0 y_0 + \gamma\sigma(x_1) y_1] + e[\sigma(x_0) y_1 + x_1 y_0]$.
2. In the basis $\{1, e\}$, this yields

$$xy = \begin{pmatrix} x_0 & \gamma\sigma(x_1) \\ x_1 & \sigma(x_0) \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}.$$

3. There is thus a correspondance

$$x = x_0 + e x_1 \in \mathcal{A} \leftrightarrow \begin{pmatrix} x_0 & \gamma\sigma(x_1) \\ x_1 & \sigma(x_0) \end{pmatrix}.$$

4. We associate to an element its *multiplication matrix*.

Cyclic algebras: matrix formulation

1. We have $xy = [x_0y_0 + \gamma\sigma(x_1)y_1] + e[\sigma(x_0)y_1 + x_1y_0]$.
2. In the basis $\{1, e\}$, this yields

$$xy = \begin{pmatrix} x_0 & \gamma\sigma(x_1) \\ x_1 & \sigma(x_0) \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}.$$

3. There is thus a correspondance

$$x = x_0 + ex_1 \in \mathcal{A} \leftrightarrow \begin{pmatrix} x_0 & \gamma\sigma(x_1) \\ x_1 & \sigma(x_0) \end{pmatrix}.$$

4. We associate to an element its *multiplication matrix*.

An involution on the algebra

- Choose the matrices A_j to be in a division algebra, so that $V_j = (\mathbf{I} - iA_j)(\mathbf{I} - iA_j)$ satisfies $\det(V_i - V_j) \neq 0$.
- To satisfy the *Hermitian* condition:

$\mathcal{A}$		$\mathcal{M}_n(L)$
x	$\leftrightarrow$	$\mathbf{X}$
$\alpha(x)$	$\leftrightarrow$	$\mathbf{X}^\dagger$
$\alpha(x) = x$	$\leftrightarrow$	$\mathbf{X}^\dagger = \mathbf{X}$

$$\alpha(x_0 + ex_1) = \bar{x_0} + e^{-1}\sigma^{-1}(\bar{x_1}).$$

An involution on the algebra

- Choose the matrices A_j to be in a division algebra, so that $V_j = (\mathbf{I} - iA_j)(\mathbf{I} - iA_j)$ satisfies $\det(V_i - V_j) \neq 0$.
- To satisfy the *Hermitian* condition:

$\mathcal{A}$		$\mathcal{M}_n(L)$
x	$\leftrightarrow$	$\mathbf{X}$
$\alpha(x)$	$\leftrightarrow$	$\mathbf{X}^\dagger$
$\alpha(x) = x$	$\leftrightarrow$	$\mathbf{X}^\dagger = \mathbf{X}$

$$\alpha(x_0 + ex_1) = \bar{x_0} + e^{-1}\sigma^{-1}(\bar{x_1}).$$

Example: 2 transmit antennas (I)

- Consider the algebra $\mathcal{A} = (\mathbb{Q}(i,\sqrt{5})/\mathbb{Q}(i),\sigma,i)$, where $\sigma : \sqrt{5} \mapsto -\sqrt{5}$.
- Let $x \in \mathcal{A}$,

$$x = x_0 + ex_1,\ x_0, x_1 \in \mathbb{Q}(i,\sqrt{5}).$$

- We compute $x = \alpha(x)$. Let $\theta = \frac{1+\sqrt{5}}{2}$. Thus, x can be written

$$x = [a_0 + \theta b_0] + e[(s(1-\theta) - t\theta) + i(t(1-\theta) - s\theta)],$$

Example: 2 transmit antennas (II)

- In matrix equations

$$\begin{aligned}\mathbf{X} &= a_0\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + s\begin{pmatrix}0 & 1-\theta-i\theta\\ i\theta+(1-\theta) & 0\end{pmatrix}\\ &\quad +b_0\begin{pmatrix}\theta & 0\\ 0 & 1-\theta\end{pmatrix} + t\begin{pmatrix}0 & -\theta+i(1-\theta)\\ -i(1-\theta)-\theta & 0\end{pmatrix}.\end{aligned}$$

We thus get a basis of 4 matrices.

Example: 2 transmit antennas (III)

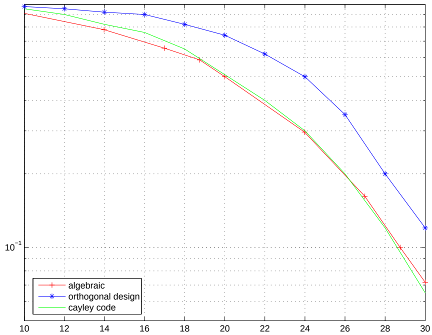


Figure: $M = 2,\ N = 2,\ R = 6,\ Q = 4$

4 transmit antennas

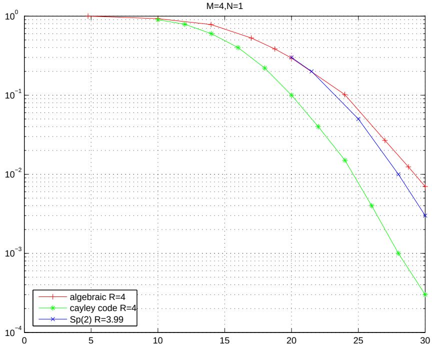


Figure: $M = 4,\ N = 1,\ R = 4,\ Q = 8$

Thank you for your attention!