A Criterion for Probabilistic Estimates

Jeremy Thorpe, Robert McEliece

April 21, 2002

Abstract

In this paper, we examine the need for a criterion with which to judge an oracle which gives estimated probability distributions. Giving A small number of motivated axioms, we arrive (??) at a single suitable measure.

1 Axioms

Consider a random variable X with outcome space $\mathbb{X}$. In addition, $\mathbb{X}$ is a metric space with distance function $D(x, x')$, which is symmetric and satisfies the triangle inequality. We are given a belief b which takes the form of a probability distribution on X and seek a quantitative measure $\triangle(b, x)$ of how accurate this belief is once the actual value $X = x$ is known. We make the following axiomatic assumptions.

Axiom 1.1

$$b(D < D') = b'(D < D') \ \forall D' \Rightarrow \triangle(b, x) = \triangle(b', x) \tag{1}$$

where $b(D < D')$ is an implicit function of x defined by:

$$b(D < D') = \sum_{x', D(x,x') < D'} b(x') \tag{2}$$

Axiom 1.2

$$E_p \triangle(p, x) \leq E_p \triangle(b, x) \ \forall b, p \tag{3}$$

Axiom 1.3

$$b(x) = 1 \Rightarrow \triangle(b, x) = 0 \tag{4}$$

2 Fundamental theorem

One such $\triangle$ which satisfies the above three axioms (?) is:

Theorem 2.1

$$\triangle(b, x) = \int_0^\infty dD' \ \log \ b(D < D') \tag{5}$$

which reduces in the case $D(x,x') = 1 - \delta(x,x')$ to the previously known

$$\triangle(b,x) = \log\, b(x) \tag{6}$$

In fact, 5 can be generalized by adding the factor $f(D')$ inside the integral, where $f(D')$ is a non-negative and non-increasing function. However, the same effect could be achieved by changing $D(x,x')$ directly. Perhaps a stronger axiom is needed to remove this freedom.

3 Future work

It must still be demonstrated that our $\triangle$ satisfies the axioms, in particular axiom 1. I can prove that it does if $D(x,x')$ can be expressed as a weighted sum of "classification metrics", by I mean there is some classifier, $c(x)$ and $D(x,x') = 1 - \delta(c(x), c(x'))$. There seems to be a concept of universal metric spaces in which every other metric space can be "embedded". If I could prove that any universal metric space could be constructed as a weighted sum (integral) of classification metrics, that would do the trick (show axiom 1 is satisfied in general).