
Problems to Hand In:

Problem 1. (Repetition codes on the AWGN channel.) Consider using the \((n, 1)\) repetition code on the AWGN channel. This code contains just two codewords, viz., \((+1, +1, \ldots, +1)\) and \((-1, -1, \ldots, -1)\). If \((x_1, x_2, \ldots, x_n)\) is the transmitted codeword, then \((y_1, y_2, \ldots, y_n)\) is received, where \(y_i = x_i + z_i\), and \((z_1, z_2, \ldots, z_n)\) are i.i.d. Gaussians with mean zero and variance \(\sigma^2\), where

\[
\sigma^2 = \left( \frac{2 \ E_b}{n \ N_0} \right)^{-1}.
\]

(a) Describe a maximum-likelihood decoding algorithm for this code.

(b) Evaluate the decoder error probability (in terms of \(E_b/N_0\)) for the decoding algorithm you found in part a.

(c) Compare this performance to that of “uncoded BPSK,” which was described in class on April 2.

Problem 2. Consider a simple “turbocode” of the following type:

\[
\begin{array}{c}
\text{u} \\
\downarrow \Pi \\
 G \\
\downarrow G \\
\text{x1} \\
\text{x2}
\end{array}
\]

Suppose \(u = (u_1, u_2, u_3, u_4)\) and \(G\) is the following \(4 \times 4\) matrix:

\[
G = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1
\end{pmatrix}.
\]

Then the overall code is an \((8, 4)\) binary linear code. However, there are \(4! = 24\) choices for the interleaver. The problem: find a “best” interleaver, i.e., one that:
(1). Ensures that the overall code has dimension 4.
(2). Maximizes the minimum distance of the code.

**Problem 3.** Consider the $(4, 2)$ binary linear code with generator matrix

$$G = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$ 

Thus the information block $(u_1, u_2)$ is encoded as $(x_1, x_2, x_3, x_4)$, where $x_1 = u_1, x_2 = u_2, x_3 = u_1 + u_2$, and $x_4 = u_1 + u_2$. Suppose that an unknown codeword is transmitted over the following DMC:

$$A \ B \ C \ D$$

$$\begin{pmatrix} 0 & 1/2 & 1/4 & 1/8 & 1/8 \\ 1/8 & 1/8 & 1/4 & 1/2 \end{pmatrix}.$$

(a) Suppose the *a priori* input distribution is

$$p^0(0) = p^0(1) = 1/2,$$

and the received word is $ABCD$. Compute the *a posteriori* probabilities for $u_1$ and $u_2$ in log-likelihood form. What is the “extrinsic information” (in log-likelihood form) for $u_1$ and $u_2$?

(b) Now suppose the *a priori* input distribution is

$$p^0(0) = 1/3, \quad p^0(1) = 2/3$$

and the received word is still $ABCD$. Answer the same questions.

**Problem 4.** In class on April 9, I proved that if $u, v$, and $x$ are vertices in a trellis, that

$$\mu_x(u, v) = \mu(u, x) \mu(x, v).$$

(a) Now suppose that $u$ and $v$ are vertices, and $e$ is an edge. Prove that

$$\mu_e(u, v) = \mu(u, x) w(e) \mu(y, v),$$

where $x$ is the initial vertex and $y$ is the final vertex of $e$.

(b) Discuss the computational savings implied by the result of part (a).