Reading:  

Problems to Hand In:

Problem 1. Consider the ensemble of LDPC codes with (left) degree profile \((\lambda_2, \lambda_3, \ldots, \lambda_N)\), and constant right degree \(a\). If the ensemble rate is \(R = 4/5\), what is the smallest possible value for \(a\)?

Problem 2. Let \(H_1 \leq \cdots \leq H_n\) be a fixed list of real numbers, and let \(p_1, \ldots, p_n\) be a corresponding probability density. Define

\[
U = \langle H \rangle_p = \sum_i p_i H_i \\
S = \langle -\log p \rangle_p = -\sum_i p_i \log p_i.
\]

For each value of \(U\) in the range \(H_1 \leq U \leq H_n\), let \(S_{\text{max}}(U)\) denote the maximum possible value of \(S\), i.e.,

\[
S_{\text{max}}(U) = \max \{-\sum_i p_i \log p_i : \sum_i p_i H_i = U\}.
\]

(a) Show that the optimizing probabilities are of the form

\[
p_i = p_i(\beta) = \frac{e^{-\beta H_i}}{Z(\beta)},
\]

where \(Z(\beta)\) (the partition function) is defined as

\[
Z(\beta) = \sum_i e^{-\beta H_i}.
\]

[Hint: If \((p_i)\) is a density such that \(\sum_i p_i H_i = U\), define \(q_i = e^{-\beta H_i}/Z(\beta)\), and apply Jensen’s inequality to the sum \(\sum_i p_i \log(q_i/p_i)\).]

(b) Show that the relationship between \(U\) and \(S_{\text{max}}\) is given parametrically by the equations

\[
U = -\frac{d}{d\beta} \log Z(\beta) \\
S_{\text{max}} = \log Z(\beta) - \beta \frac{d}{d\beta} \log Z(\beta),
\]

where the parameter \(\beta\) (the inverse temperature) runs from \(-\infty\) to \(+\infty\).

(c) For \((H_1, H_2, H_3) = (1, 2, 3)\), plot the function \(S_{\text{max}}\) as a function of \(U\).